A Dynamic Game on Renewable Natural Resource Exploitation

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1 Introduction

This paper studies firms' exploitation of renewable natural resources in a dynamic setting. We present a differential game to examine oligopolistic firms' harvesting behavior in a continuous time infinite horizon model. Although differential games have been widely used in the economics literature, it is well known that finding feedback equilibrium is extremely difficult except linear-quadratic games. Nevertheless, for models of exploitation of renewable natural resources, we need to analyze games that are not linear-quadratic since reasonable growth functions regarding the stock of natural resources are not linear. In this paper, we present a differential game model that is not linear-quadratic and derive Markov feedback equilibrium for the game. One prominent feature of the model is that demand for harvest of a natural resource is assumed to depend upon the stock of the natural resource.

In our model, we consider two settings regarding firms' harvesting decisions. One setting is the case where the firms make their decision noncooperatively and the other is the one where the firms can cooperate. For the noncooperative case, the firms are assumed to undertake Cournot competition in the product market.

In the context of fishery economics, Levhari and Mirman (1980) examine competition of harvesting in a dynamic setting. They study utility maximization, but do not consider profit maximizing firms. In this paper, we will analyze a dynamic oligopolistic model of a renewable natural resource that is not linear-quadratic. We will examine feedback strategies and under certain conditions, derive Markov perfect equilibrium. We will also discuss the existence and the multiplicity of open-loop Nash equilibrium. In general, there can be multiple open-loop Nash equilibria.

The paper is organized as follows. In Section 2, we will describe the basic model. In Section 3, we examine Markov feedback strategies and derive Markov perfect equilibrium. In Section 4, we will examine open-loop Nash equilibrium for games both under the case of noncooperation and under the case of cooperation. In Section 5, we discuss the effects of taxation on equilibrium exploitation of the renewable natural resource. Section 6 concludes.

2 The Model

There are n firms in the industry, indexed by $i \in I = \{1, \cdots, n\}$. The firms harvest a renewable natural resource and sell them in the product market. We assume that the firms engage in Cournot competition in the product market.

Let $g(X)$ be the growth function of the natural resource. We assume

$$g(0) = 0, g(\tilde{X}) = 0 \text{ for some } \tilde{X} > 0, g^{'}(0) > 0, \text{ and } g^{''}(X) < 0.$$

Then the change in the stock at time t, $\dot{X}$, is given by

$$\dot{X} = \frac{dX}{dt} = g(X) - \sum_{i=1} x_i. \tag{1}$$

The initial stock of the natural resource is denoted

$$X(0) = X_0.$$

For each firm, the constant unit cost of harvesting is given by $C(X)$. We assume

$$\frac{\partial C}{\partial X} < 0 \text{ and } \frac{\partial^2 C}{\partial X^2} > 0.$$

Let the inverse demand function be given by

$$p = f(\sum_{i=1} x_i, X),$$

where p is the price of the product, x_i the harvest as well as the product of firm i, and X the stock of the renewable natural resource.

Note that demand for the product depends on the stock of the natural resource. In other words, we assume that consumers are concerned with the environment, that is, the stock of the natural resource when deciding their demand.

We also assume

$$f^{''} x_i + 2f^{'} < 0.$$

where $f' \equiv \frac{\partial f}{\partial x_i}$.

The objective of each firm is to maximize the discounted sum of its profits over an infinite time horizon. Let r be a common discount rate. Then the objective of firm i is given by

$$J_i = \int_0^\infty [px_i - C(X)x_i]\, e^{-rt} dt. \tag{2}$$

3 Markov Perfect Equilibrium

In this section, we consider feedback strategies. In particular, we examine Markov feedback strategies. Feedback strategies and Markov perfect equilibrium are defined as follows.

Definition 1 The feedback strategy space for firm i is the set

$$S_i^F = \{x_i(X,t) \text{ is continuous in } (X,t),\ x_i(X,t) \geq 0, \text{ and } X \geq 0\}.$$

Definition 2 A Markov perfect equilibrium is a pair of feedback strategies (x_i^*, x_{-i}^*) such that for each $i \in I$,

$$J_i(x_i^*, x_{-i}^*) \geq J_i(x_i, x_{-i}^*) \text{ for every } x_i \in S_i^F.$$

In order to derive a closed form solution, we assume that the inverse demand function is given by

$$p = f(\sum_{i=1} x_i, X) = \frac{a}{X} - \frac{b\sum_{i=1}^n x_i}{X^2},\ a > 0,\ b > 0. \tag{3}$$

We also assume that the cost function takes the following form:

$$C(X) = \frac{\bar{c} - \gamma \ln X}{X},\ \bar{c} > 0,\ \gamma > 0. \tag{4}$$

Note that $C'(X) > 0$ and $C''(X) < 0$.

Furthermore we assume that the growth function of the stock level of the natural resource takes the following form:

$$g(X) = X(\alpha - \beta \ln X),\ \alpha > 0,\ \beta > 0. \tag{5}$$

In what follows, we consider the case of two firms, i.e., we assume $n = 2$. Then we have the following theorem.

Theorem 3 There exists a Markov perfect equilibrium given by

$$x_i^* = \frac{a - \bar{c} - F + (\gamma - G)\ln X}{3b} \bullet X,$$

where G and F are given in (16) and (17), and $3b\beta - 2\gamma + 2G < 0$.

Proof: First let $Y \equiv \ln X$ and $y_i \equiv \frac{x_i}{X}$.
Then the objective function of firm i may be rewritten as

$$\int_0^\infty [a - (y_i + y_j) - (\bar{c} - \gamma Y)] y_i e^{-rt} dt. \tag{6}$$

Also the growth function may be rewritten as

$$\frac{\dot{X}}{X} = \alpha - \beta \ln X - \frac{\sum_{i=1}^n x_i.}{X}.$$

That is

$$\dot{Y} = \alpha - \beta Y - (y_i + y_j),\ i, j = 1,\ 2,\ i \neq j. \tag{7}$$

Let $V^i(X)$ be the value function for firm i. Then the system of Hamilton-Jacobi-Bellman equation becomes

$$rV^i(Y) = \max_{y_i} \left\{ [a - b(y_i + y_j) - (\bar{c} - \gamma Y)] y_i + \frac{dV^i(Y)}{dY}[\alpha - \beta Y - (y_i + y_j)] \right\} \tag{8}$$

Solving the maximization problem of the right hand side of (8) yields

$$y_i^* = \frac{a - c(Y) - \frac{dV^i(Y)}{dY}}{3b},$$

where $c(Y) \equiv \bar{c} - \gamma Y$.

Now we assume that the value function is symmetric. Suppose that the value function takes the following form:

$$V(Y) = E + FY + \frac{1}{2}GY^2. \tag{9}$$

Then

$$\frac{dV(Y)}{dY} = F + GY. \tag{10}$$

Thus we get

$$y^* = \frac{a - c(Y) - \{F + GY\}}{3b}. \tag{11}$$

Substituting (9), (10) and (11) into (8), we get

$$\begin{aligned} & r\left(E + FY + \frac{1}{2}GY^2\right) \qquad (12) \\ = & \left(a - 2\left\{\frac{a - c(Y) - F - GY}{3b}\right\} - c(Y)\right)\left\{\frac{a - c(Y) - F - GY}{3b}\right\} \\ & + (F + GY)\left(g(Y) - 2\left\{\frac{a - c(Y) - F - GY}{3b}\right\}\right). \end{aligned}$$

Let

$$L \equiv a - \bar{c} - F \quad \text{and} \quad M \equiv \gamma - G.$$

The equation (12) must hold for any Y, and hence we have

$$\frac{1}{2}rG + 2bM^2 - \gamma M - \beta G + 2GM = 0, \tag{13}$$

$$rF - (a - \bar{c})M + 4bLM - \gamma L - \beta F - \alpha G + 2GL + 2FM = 0, \tag{14}$$

and

$$rE - (a - \bar{c})L + 2bL^2 - \alpha F + 2FL = 0. \tag{15}$$

It follows from (13), (14) and (15) that we obtain

$$G = \frac{(9br - 18b\beta + 10\gamma) \pm \sqrt{(9br - 18b\beta + 10\gamma)^2 - 64\gamma^2}}{16}, \tag{16}$$

$$F = \frac{(a - \bar{c})(2\gamma - 5G) + 9b\alpha G}{9b(r - \beta) + 5r - 8G}, \tag{17}$$

and

$$E = \frac{1}{r}\left[(a - \bar{c})L - 2bL^2 + \alpha F - 2FL\right]. \tag{18}$$

Also

$$M = \frac{3b\beta + \gamma - \frac{3}{2}br \pm \sqrt{(3b\beta + \gamma - \frac{3}{2}br)^2 + 8b\gamma(r - 2\beta)}}{8b}. \tag{19}$$

Next substituting (11) into (7) yields

$$\dot{Y} - \{\beta - \frac{2\gamma}{3b} - \frac{2G}{3b}\}Y + \frac{2}{3b}\{a - \bar{c} - F\} - \alpha = 0. \tag{20}$$

A particular solution to the differential equation (20) is

$$Y = \frac{2(a - \bar{c} - F) - 3b\alpha}{3b\beta - 2\gamma + 2G}.$$

Then the solution of (20) is

$$Y(t) = Y + (Y_0 - Y)e^{\{\frac{3b\beta - 2\gamma + 2G}{3b}\}t}. \tag{21}$$

We must have $3b\beta - 2\gamma + 2G < 0$ in order that this state trajectory is asymptotically stable.

We note that when $\beta > 0$, we must have $M > 0$. Therefore $G < \gamma$.

4 Open-loop Nash Equilibrium

In this section, we consider open-loop strategies. Open-loop strategies and open-loop Nash equilibrium are defined as follows.

Definition 4 The open-loop strategy space for firm i is the set

$$S_i = \{x_i(t) : x_i(t) \text{ is piecewise continuous and } x_i(t) \geq 0 \text{ for every } t\}.$$

Definition 5 An open-loop Nash equilibrium is an open-loop strategy selection $x^* = (x_i^*, x_{-i}^*)$ such that for each $i \in I$,

$$J_i(x_i^*, x_{-i}^*) \geq J_i(x_i, x_{-i}^*), \text{ for } \forall x_i \in S_i.$$

First, we consider the case where the firms choose harvesting strategies noncooperatively.

For firm i, the sum of its discounted profits is given by

$$\begin{aligned} J_i &= \int_0^\infty [p x_i - C(X) x_i]\, e^{-rt} dt \qquad (22) \\ &= \int_0^\infty [p - C(X)]\, x_i e^{-rt} dt. \end{aligned}$$

The current value Hamiltonian for firm i is then given by

$$H_i = [p - C(X)]\, x_i + \lambda_i \left(g(X) - \sum x_i \right), \qquad (23)$$

where λ_i is a costate variable.

Let $f_X \equiv \frac{\partial f}{\partial X}$. The necessary conditions for an open-loop Nash equilibrium are

$$\frac{\partial H_i}{\partial x_i} = f' x_i + f - C(X) - \lambda_i = 0, \qquad (24)$$

$$\begin{aligned} \dot{\lambda}_i &= r\lambda_i - \frac{\partial H_i}{\partial X} \qquad (25) \\ &= r\lambda_i - \left(f_X \cdot x_i - C'(X) x_i + \lambda_i g'(X) \right) \\ &= \left(r - g'(X) \right) \lambda_i + (C'(X) - f_X) x_i, \end{aligned}$$

and

$$\lim_{t \to \infty} e^{-rt} \lambda_i = 0. \qquad (26)$$

Summing equation (24) over i, we get

$$f' \sum x_i + n\,(f - C(X)) - \sum \lambda_i = 0. \qquad (27)$$

Let $Q = \sum x_i$. Then equation (27) may be rewritten as

$$f' Q + n\,(f - C(X)) - \sum \lambda_i = 0. \qquad (28)$$

Differentiate (28) with respect to time, we have

$$f''(\dot{Q} + \dot{X}) Q + f' \dot{Q} + n \left(f'(\dot{Q} + \dot{X}) - C' \dot{X} \right) - \sum \dot{\lambda}_i = 0. \qquad (29)$$

Recall that

$$\begin{aligned}\dot{X} &= g(X) - \sum x_i \\ &= g(X) - Q\end{aligned}$$

and

$$\dot{\lambda}_i = \left(r - g^{'}(X)\right)\lambda_i + (C^{'}(X) - f_X)x_i.$$

Then we have

$$\begin{aligned}\sum \dot{\lambda}_i &= \left(r - g^{'}(X)\right)\sum \lambda_i + (C^{'}(X) - f_X)\sum x_i \\ &= \left(r - g^{'}(X)\right)\sum \lambda_i + (C^{'}(X) - f_X)Q.\end{aligned} \tag{30}$$

Thus (29) may be rewritten as

$$\begin{aligned}& f^{''}(\dot{Q} + g(X) - Q)Q + f^{'}\dot{Q} \\ & +n\left(f^{'}(\dot{Q} + g(X) - Q) - C^{'}(g(X) - Q)\right) \\ & -\left(r - g^{'}(X)\right)\sum \lambda_i \\ -(C^{'}(X) - f_X)Q &= 0.\end{aligned} \tag{31}$$

Hence we have

$$\begin{aligned}& \dot{Q}\left(f''Q + (n+1)f'\right) - f''Q^2 \\ & -\{-f_X + (n-1)C' + f''g - nf^{'} + (r - g^{'})f'\}Q \\ & -(r - g^{'})n\,(f - C) \\ +n(f'g - C'g) &= 0\end{aligned} \tag{32}$$

It follows from (32) that

$$\begin{aligned}& \dot{Q} \\ = & \frac{f''Q^2 - f_X Q + \{(n-1)C' + f''g - nf^{'} + (r - g^{'})f'\}Q}{f''Q + (n+1)f'} \\ & + \frac{(r - g^{'})n\,(f - C) - n(f'g - C'g)}{f''Q + (n+1)f'}.\end{aligned} \tag{33}$$

Recall that the inverse demand function is

$$p = f(Q,\, X) = \frac{a}{X} - \frac{bQ}{X^2},\ a > 0, b > 0.$$

Then for the locus of $\dot{Q} = 0$, we have

$$\{(n-1)C' - f_X - nf' + (r-g')f'\}Q + (r-g')n\,(f - C) - n(f'g - C'g) = 0. \quad (34)$$

Let the left hand side of (34) be $\Psi(X, Q)$.

First note that there exists $\check{X} > 0$ such that $\Psi(\check{X}, 0) = 0$. Note also that, by the implicit function theorem, we have

$$\frac{dQ}{dX} = -\frac{\frac{\partial \Psi}{\partial X}}{\frac{\partial \Psi}{\partial Q}} \gtreqless 0.$$

In what follows, we assume that $\frac{dQ}{dX} > 0$ and that $Q = \Phi(X)$ solves (34). Then, at the steady state, i.e., $\dot{Q} = 0$ and $\dot{X} = 0$, we see that there exists an open-loop Nash equilibrium harvest Q^* and the stock level X^* such that $\Phi(X^*) = g(X^*)$.

Note that $g'(X^*) \gtreqless 0$. It may be possible that there exist more than one equilibrium.

We have so far analyzed the case where the firms choose their strategies noncooperatively. Now we consider the case where the firms cooperate when they harvest the natural resources.

The objective function in this case is given by

$$\sum_i \int \Big[f(\sum x_i, X) - C(X) \Big] x_i e^{-rt} dt. \quad (35)$$

The current value Hamiltonian is then given by

$$K_i = \sum_i \Big[f(\sum x_i, X) - C(X) \Big] x_i + \xi \Big(g(X) - \sum x_i \Big)', \quad (36)$$

where ξ is a costate variable.

The necessary conditions for an open-loop Nash equilibrium are

$$\frac{\partial K_i}{\partial x_i} = \sum f' x_i + f - C(X) - \xi = 0, \quad (37)$$

$$
\begin{aligned}
\dot{\xi} &= r\xi - \frac{\partial K_i}{\partial X} \\
&= r\xi - \left(f_X - C'(X) \sum x_i + \xi g'(X) \right) \\
&= \left(r - g'(X) \right) \xi + (C'(X) - f_X) \sum x_i,
\end{aligned}
\tag{38}
$$

and

$$
\lim_{t \to \infty} e^{-rt} \xi = 0. \tag{39}
$$

Differentiate (37) with respect to time, we obtain

$$
f''(\sum \dot{x}_i)(\sum x_i) + f'(\sum \dot{x}_i) - C'\dot{X} - \dot{\xi} = 0. \tag{40}
$$

Substituting (37) and (38) into (40), we have

$$
f''\{\dot{Q}+(g-Q)\}Q+f'\dot{Q}-C'(g-Q)-\left(r-g'\right)\{f'Q+f-C\}+(f_X-C')Q=0. \tag{41}
$$

It follows from (41) that we have

$$
\dot{Q} = \frac{f''Q^2 - f_X Q + \{\left[r-g'\right]f' - f''g\}Q + \left[r-g'\right](f-C) + C'g}{f''Q+f'}. \tag{42}
$$

Recall that the inverse demand function is given by the following form,

$$
f(Q,\, X) = \frac{a}{X} - \frac{bQ}{X^2}.
$$

Then for the locus of $\dot{Q} = 0$,

$$
f_X Q - \left(r - g'\right) f'Q - \left(r - g'\right)(f - C) - C'g = 0. \tag{43}
$$

Let the left hand side of (43) be $\psi(X, Q)$.
Note also that

$$
\frac{dQ}{dX} = -\frac{\frac{\partial \psi}{\partial X}}{\frac{\partial \psi}{\partial Q}} \gtreqless 0.
$$

In what follows, we assume that $\frac{dQ}{dX} > 0$ and that $Q = \phi(X)$ solves (43). Then, at the steady state, i.e., $\dot{Q} = 0$ and $\dot{X} = 0$, we see that there exists an open-loop Nash equilibrium harvest Q^* and the stock level X^* such that $\phi(X^*) = g(X^*)$.

Note that $g'(X^*) \gtreqless 0$. It may be possible that there exist more than one equilibrium.

5 Taxation

In this section, we will examine effects of taxation on equilibrium harvest. We consider a specific tax whose rate depends upon the stock of the natural resource. Let $\frac{\theta}{X}$ be a tax rate, and $\theta \geq 0$. Thus, given θ, the smaller the stock of the natural resource, the higher the tax rate. The objective function of each firm is given by

$$\int_0^\infty \left(p - C(X) - \frac{\theta}{X} \right) x_i e^{-rt} dt. \tag{44}$$

For Markov perfect equilibrium, we have

$$y^* = \frac{a - c(Y) - \theta - \{F + GY\}}{3b}.$$

Thus we get

$$\frac{\partial y^*}{\partial \theta} < 0.$$

Therefore if the tax rate increases, then the equilibrium harvest rate will decrease.

For examining an open-loop Nash equilibrium, the current value Hamiltonian becomes

$$T_i = \int_0^\infty \left(p - C(X) - \frac{\theta}{X} \right) x_i e^{-rt} dt + \mu_i [g(X) - \sum_{i=1} x_i], \tag{45}$$

where μ_i is a costate variable.

Then the necessary conditions for an open-loop Nash equilibrium are

$$f' x_i + f - C - \frac{\theta}{X} - \mu_i = 0, \tag{46}$$

$$
\begin{aligned}
\dot{\mu}_i &= r\mu_i - \frac{\partial T_i}{\partial X} \qquad (47) \\
&= r\mu_i - \left((f_X + \frac{\theta}{X^2} - C'(X))x_i + \mu_i g'(X) \right) \\
&= \left(r - g'(X) \right) \mu_i - (f_X + \frac{\theta}{X^2} - C'(X))x_i,
\end{aligned}
$$

and

$$
\lim_{t\to\infty} e^{-rt}\mu_i = 0. \qquad (48)
$$

Then for the locus of $\dot{Q} = 0$, we have

$$
\begin{aligned}
&\{(n-1)C' + \frac{\theta}{X^2} - f_X - nf' + (r - g')f'\}Q \qquad (49) \\
&+(r - g')n \left(f - C - \frac{\theta}{X} \right) - n(f' - C')(g - \frac{\theta}{X^2}) = 0.
\end{aligned}
$$

Let the left hand side of be $\Omega(X, Q)$.

$$
\begin{aligned}
\frac{dQ}{dX} &= -\frac{\frac{\partial \Omega}{\partial X}}{\frac{\partial \Omega}{\partial Q}} \qquad (50) \\
&= -\frac{\frac{\partial \Psi}{\partial X} - \frac{2\theta Q}{X^2} + g''n(\frac{\theta}{X})}{\frac{\partial \Psi}{\partial X} + \frac{\theta}{X^2}} \\
&\quad -\frac{(r - g' + f - C'')\frac{n\theta}{X^2} - 2n(f' - C')\frac{\theta}{X^3}}{\frac{\partial \Psi}{\partial X} + \frac{\theta}{X^2}}.
\end{aligned}
$$

6 Conclusion

In this paper, we have studied firms' exploitation of renewable natural resources in a dynamic setting. We have constructed a differential game to examine oligopolistic firms' harvesting in a continuous time infinite horizon model. For models of exploitation of renewable natural resources, we have analyzed a game that is not linear-quadratic and derived Markov perfect equilibrium for the game.

In the model, we have considered two settings regarding firms' harvesting decisions. One setting is the case where the firms make their decision noncooperatively and the other is the one where they can cooperate. We have also analyzed open-loop Nash equilibrium for the games both under the case of noncooperation and under that of cooperation. We have also examined the effects of taxation on equilibrium exploitation of the natural resource.

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