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The Mathematical Decomposition of the Transactions Velocity of Money *

Hakun Kim **

Professor of Economics
Department of Economics
Yonsei University
Seoul, 120-749, Korea

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(i) Name of the Author : Hakun KIM
(ii) Title of the Paper : The Mathematical Decomposition of the Transactions Velocity of Money
(iii) Mailing Address : Hakun KIM
Professor of Economics
(Office) Department of Economics
College of Economics and Business
Yonsei University
134 Shinchon-dong, Seodaemun-ku
Seoul, 120-749, Korea
Tel) 02-2123-2472
email : hakun@base.yonsei.ac.kr

(Home) # 1101 (House Number)
101 Dong (Building Number)
Namgazwa Hyundai Apartment Complex
376 Namgazwa-dong , Seodaemun-ku
Seoul, 120-758, Korea
Tel and Fax) 02-376-2445

(iv) Abstract : This paper is an attempt to decompose the average transactions velocity of money into two or more individual velocities. When the economy-wide velocity is expressed as a weighted average of two disaggregated velocities, this provides an equation with two unknowns. The additional equation can be created from the concept of two versions of the exchange equations; the Fisherian and Cambridge equation. The former represents the Fisherian problem, while the latter the Marshallian problem. Their integration furnishes us with the second equation to solve the system.

(v) Key Words : velocity indifference curve, iso-velocity line, transactions time indifference curve, iso-transaction time line, law of equal marginal velocities per dollar, Fisherian space, Marshallian space, Hicks strong parallel movement effect, Hicks strong substitution effect, Hicks weak parallel movement effect, Hicks weak substitution effect

(vi) JEL Classification : E4, G2

I. Introduction

The purpose of this paper is to decompose the economy-wide average transactions velocity of money into different sectoral velocities. It is true that sectoral money and its velocity are determined in the general equilibrium framework, but one may find the framework not useful for the decomposition unless he/she knows the sectoral money supply mechanism. The decomposition adopted in this paper is, therefore, a purely mathematical process without any economic assumption.

As Keynes (1930) points out, the importances of finding disaggregate velocities and monies is clear. For example, the sectoral velocities might fluctuate but offset each other leaving the weighted average velocity stable, and vice versa. Observing the latter only would lead to the erroneous conclusion regarding the underlying sectoral activities. One would think of an immediate application to a country's annual series of real and financial transactions. From the macroeconomic point of view, the knowledge of financial and real velocities over time would be critically important in explaining the episode like the great velocity decline. The velocity series for real income process would provide clues on the short-run and long-run effects of monetary policy.

Despite its importance, the decomposition has no place in modern economic analysis even though it attracted some interest in the decades before 1940. As Cramer (1989) observes, "very few people care." In recent years, Humphrey (1993) is the only one who shows in the theoretical level his interest in it by introducing Petty's (1662) decomposition, while Selden (1961), McGouldrick (1962), Garvey (1969), and Ireland (1991) discuss sectoral velocities in the empirical level.[1] Nevertheless, the decomposition problem will be a subject of high-level theory and policy concern for at least three main reasons.

First, much of the literature has discussed aggregation problems of micro-level magnitudes. On the contrary, the decomposition of the transaction velocity of money presented in this paper would provide the general method for the reverse process that any aggregate measure may be disaggregated into sub-level components with limited information.

Second, one may find the decomposition useful in analysing the effect of the newly born euro on each member country's holdings and its velocity after the European monetary union is fully launched. Miller (1979) gives us the impression that identifying U. S. regional monies and their velocities would assist the U. S. monetary authorities to allocate the limited amount of available credit more effectively.

Third, evolution of private and public payment arrangement due to the recent development of electronic technologies has very different impacts on different sectors and regions in substantial levels. Weinberg (1994) discusses this change, saying, "In check processing, renewed growth has occurred in the activities of clearinghouses on

1) I am indebted these references to Dr. Thomas Humphrey in the Federal Reserve Bank of Richmond.

local, regional and, most recently, national levels" (p.2). It was only 1984 when Tobin (1984) reports that the GNP velocity of the money stock in the US is 6 or 7 per year. If financial transactions are included, the turnover may be 20 or 30 per year. But demand deposits turn over 500 times a year, 2500 times in New York banks, indicating that most transactions are financial in nature. Compare these figures in 1984 with those of the Bank for International Settlements report in 1996. In the United States, roughly 220 million market transactions are made without cash daily, with a total dollar value of $1.6 trillion; a tremendous increase in velocity in credit transactions. In 10 years, improvements in computer and telecommunications technologies facilitate the speed and reliability particularly in credit transactions area.

This paper has as its starting point Fisher's 1911 transactions velocity function. We could divide all transactions into those related to the level of national income and other financial transactions. The equation may be expressed as :

$$MV = T^Y + T^F \tag{1}$$

where M and V are nominal stock of money and its velocity, and T^Y and T^F are nominal values of real and financial transaction volumes, respectively. The emphasis on the real economic activities would transform the right-hand side into just the nominal income and the transactions velocity in the left-hand side into the income velocity.

The income velocity used in the literature, nominal income divided by a monetary aggregate, is simply the demand for money, which in turn originates from the exchange equation. Restricting the transactions to the income-generating process, economists have transformed the equation into the demand for real balances. When the restriction remains valid, that is, when there is a close and stable relation between the total volume of overall transactions and that of income-generating ones, the behavior of income velocity would depict the underlying theory of the exchange equation.

The theoretical underpinning for advocating active monetary policies lies in the belief that the quantity of money bears a predictable relation with income and interest rates. The US's experiences in the 80's such as "the great velocity decline" and "missing money", however, have caused a havoc in the field of monetary macroeconomics. On one side, economists have enlisted a series of explanation defending the viability of the stable money demand or the predictable income velocity. The creation of interest-bearing demand deposits, the change in monetary operating procedures, and the occurrences of various financial innovations are a few examples. On the other hand, there has been a steady increase in the number of empirical studies demonstrating the nonstationarity of income velocity, particularly that of M1, hence casting a serious doubt on the efficacy of monetary policies.

The left-hand side of equation (1) has also been studied in detail. Keynes (1930), following the earlier works by des Essars (1895), Fisher (1911), and Snyder (1924), bisected MV into $M^Y V^Y$ and $M^F V^F$ where the superscripts denote different types of deposits (money) and their velocities as in equation (1). However, he rejected the idea of a constant relationship between the total stock of money M and T^Y and suggested a disaggregated analysis of income and financial velocity of money. He writes

> It is important, therefore, to distinguish between the 'average' velocity of money in a variety of uses and the 'true' velocity of money in a particular use meaning by the latter the ratio of the volume of a particular type of transactions to the quantity of money employed in them, for fluctuations in 'average' velocities may be due, not to fluctuations in 'true' velocities, but to fluctuations in the relative importance of different types of transactions. (Keynes, 1930, vol. II, p.38, quotations in original).

This statement may be expressed as a simple functional form by adopting Keynes' definition of the 'true' velocity V^Y and V^F as $\frac{T^Y}{M^Y}$ and $\frac{T^F}{M^F}$, respectively, and introducing them into equation (1). Then, we have

$$MV = M^Y V^Y + M^F V^F \tag{2}$$

which is equivalent to

$$V = m^Y V^Y + (1 - m^Y) V^F \tag{3}$$

where $m^Y = \frac{M^Y}{M}$. In this sense, Keynes' statement amounts to the definition of the 'average' velocity V expressed as the arithmetic mean of the 'true' velocity V^Y and the 'true' velocity V^F.

The question is, of course, how to divide the 'average' velocity V into the 'true' velocities. This division needs to be economically meaningful and statistically operative. Keynes' interest in the decomposition developed later into liquidity preference theory in which motives for holding real balances like speculative and transaction demand are conceptually important and attractive, but not available in each period for empirical analysis. Divisia index of Barnett (1980) and MQ index of Spindt (1985) are two distantly related attempts to disaggregate the average use of money. However, their objective was to devise an alternative measure of weighted average of money, ranging from *M1* to the total liquidity rather than individual money concepts within the aggregate measure of money and their velocities. Despite these difficulties, there are studies on the velocity of cash balances (Fisher 1909, Snyder 1924, Laurent 1970, Cramer and Reekers 1976, Cramer 1981, Boeschoten and Fase 1984). Cramer (1989) brilliantly summarizes the main troubles of the decomposition.

This paper is an attempt to decompose the average velocity into two or more individual velocity sequences. It will be shown that the decomposition is a purely mathematical process without any economic implication and/or assumption. In practice, the decomposition depends on the nature and availability of actual transaction volumes. For example, it could be real versus financial transactions. We have approximate measures of actual income transactions and financial transactions of an economy during a period of time. This provides an equation with two unknowns, two sectoral velocities. The key identifying these two unknowns lies in the additional equation that can be generated from the concept of two versions of the exchange equation, the Fisherian and Cambridge equations. The former generates the Fisherian problem, while the latter the Marshallian problem. In essence, this adds another independent equation. The complication arises from the intricate additive relationships among transaction measures and the velocities. The algebraic derivation for the unique solution for individual velocities is simply the bifurcation process of the average velocity. In fact, this process can continue to any sub-level of transaction in a straightforward manner.

II. The Exchange Equations in the Fisherian Space

1. Basic Definitions

Decomposing an economy into two sectors (or regions or industries, etc.), we start with an identity *in each period*

$$A \equiv A_1 + A_2 \qquad (4)$$

where A , A_1 , and A_2 are total and two sectoral nominal transaction volumes *in each period*, respectively, which are observable. From the quantity of money aspect we may define

$$M \equiv M_1 + M_2 \qquad (5)$$

where M is the total stock of money, M_1 is the quantity of money employed in the first sector, and M_2 is the quantity of money employed in the second sector. Following Keynes' definition (1930), we would distinguish between the 'average' velocity and the 'true' velocities

$$V \equiv \frac{A}{M}, \qquad V_1 \equiv \frac{A_1}{M_1}, \qquad V_2 \equiv \frac{A_2}{M_2} \qquad (6)$$

where V_1 is the first sector's 'true' velocity, V_2 is the second sector's 'true' velocity, and V is the economy-wide overall 'average' velocity. All velocities are measured as the number of turnovers *per unit of time.* It may be convenient to name the expression (6) the Fisherian problem, where the i sector's money M_i and the i sector's transaction volume A_i are determined, as endogenous variables, in the general equilibrium system.

2. The Iso-velocity line

Substituting the Fisherian exchange equation (6) into the identity (4) would produce a straight line as follows :

$$V = mV_1 + (1-m)V_2 \tag{7}$$

where $m = \frac{M_1}{M}$, the first sector's money ratio. Equation (7) is the same form as equation (3) that Keynes suggested with the statement that the economy's overall 'average' velocity is the weighted 'arithmetic mean' of each sector's 'true' velocity, V_1 and V_2 , where the weight m should be, by definition, bounded as

$$1 > m > 0 \tag{8}$$

The geometrical expression of equation (7) is shown in Figure 1, which represents the Fisherian space. Not only V but also m is constant along the straight line V. Therefore, the location and slope of equation (7) depend upon V and m so that we would use the symbol $V[m]$ for the straight line and call it the iso-velocity line of V.

Here is the place of <Figure 1>

3. The Velocity Indifference Curve

Substituting (6) into (5) yields the second Fisherian equation :

$$\frac{1}{V} = \frac{a}{V_1} + \frac{1-a}{V_2} \tag{9}$$

where $a = \frac{A_1}{A}$ the first sector's transaction ratio. This equation, a CES functional form with the elasticity of substitution being 1/2, forms the rectangular hyperbola V in Figure 1, an indifference curve of velocity representing a constant level of V in the (V_1, V_2) plane. Not only V but also a is constant along the indifference curve V.

Therefore, the location and curvature of the indifference curve depend upon V and a so that we would denote it as $V[a]$ and name it the velocity indifference curve of V. It is interesting to note that equation (9), unlike equation (7), says that the economy's overall 'average' velocity V is the weighted 'harmonic mean' of the sectoral 'true' velocities, where the weights are a and $1-a$, respectively. The coordinates of its center are defined by $O_V = [\, aV,\ (1-a)V\,]$.

The velocity indifference curve $V[a]$ meets the iso-velocity line $V[m]$ at point F and point E, defined by

$$E = (V,\ V), \qquad\qquad F = \left(\frac{a}{m}V,\ \ \frac{1-a}{1-m}V\right) \tag{10}$$

These two solutions may be obtained by substituting the 'harmonic mean' in (9) into the 'arithmetic mean' in (7). Since it is likely that $a \neq m$ in general, the trivial solution E, situated on the 45 degree line in Figure 1, would be ruled out and the solution F is the 'true' solution. Then the 'true' velocities at point F are characterized by

$$V_1 \neq V_2 \qquad\qquad \text{as} \quad a \neq m \tag{11}$$

We now suppose that the economy is at point F, the precise location of which we are going to identify in the following by investigating the key properties of point F.

4. The Law of Equal Marginal Velocities

The slope of the straight line connecting the center O_V and the point F is equal to the absolute value of the slope of the slope, $(-p)$, of a straight line tangent to the indifference curve V at point F, which, in turn, must equal the ratio of the marginal velocities[2] at that point as follows :

$$-p = \left.\frac{dV_2}{dV_1}\right|_F \tag{12}$$

Then, at point F the following condition holds from (9) and (12) :

$$p = \left.\frac{a}{1-a}\left(\frac{V_2}{V_1}\right)^2\right|_F \tag{13}$$

2) The marginal velocity with respect to time is sometimes called the accelerator.

This is the equality of the rate of marginal velocity substitution of V_1 for V_2 and the absolute value of the slope p at point F. If one would interpret the slope, $(-p)$, as the ratio of the transaction costs of the two velocities, $\left(-\frac{p_1}{p_2}\right)$, another way of stating (13) is

$$\frac{\left(\frac{a}{V_1^2}\right)}{p_1} = \frac{\left(\frac{1-a}{V_2^2}\right)}{p_2} \tag{14}$$

It follows from (14) that (13) may be called the law of equal marginal velocities per dollar at point F. Since point F may be defined by the slope, p, we use the notation $F[p]$ as an indication of the point. It may be instructive to note that we are not necessarily required the optimization principle to obtain (13), but admit it as the mathematical definition of the slope at a point.

5. The Parallel Movement Effect

Investigating the properties of point F only is not sufficient for the decomposition. We need the comparative statics method that is the investigation of changes in a system from one situation of equilibrium to another without regard to the transitional process.[3] The movement to point F in the above may be considered as a composition of the parallel movement effect [4] and the substitution effect in the comparative statics sense. Suppose that the economy moves from a situation characterized by transactions $\{C, C_1, C_2\}$ to another situation defined by transactions $\{A, A_1, A_2\}$ analyzed in the above. This arrangement gives us another definition of transactions in the comparative statics manner:

$$C = C_1 + C_2 \tag{15}$$

In order to exchange these transaction volumes, the economy needs the quantity of money in the economy-wide as well as in the sectoral levels:

$$N = N_1 + N_2 \tag{16}$$

It is also true that the first sector's money ratio $\frac{N_1}{N}$ and the first sector's transaction

3) Samuelson (1965), p.8.
4) This effect is equivalent to the income effect in Hicks sense.

ratio $\frac{C_1}{C}$ are determined in the general equilibrium system. It is the variety of velocities that would convert the quantity of money in (16) to the transaction volumes in (15) such that

$$Z \equiv \frac{C}{N}, \qquad Z_1 \equiv \frac{C_1}{N_1}, \qquad Z_2 \equiv \frac{C_2}{N_2} \tag{17}$$

Substituting the exchange equations (17) into (15) gives us a new iso-velocity line $Z[n]$ defined by

$$Z = nZ_1 + (1-n)Z_2 \tag{18}$$

where $n = \frac{N_1}{N}$ the first sector's money ratio. This is another 'arithmetic mean' denoted by $Z[n]$. Figure 1 compares the iso-velocity line $Z[n]$ with the iso-velocity line $V[m]$.

In the similar manner, substituting (17) into (16) gives us another new indifference curve defined by

$$\frac{1}{Z} = \frac{c}{Z_1} + \frac{1-c}{Z_2} \tag{19}$$

where $c = \frac{C_1}{C}$ the first sector's transaction ratio where $c \neq n$ and $1 \neq c + n$ in general. This is another 'harmonic mean' denoted by $Z[c]$. Figure 1 depicts equation (19) as the rectangular hyperbola $Z[c]$, the coordinates of whose center is O_Z = $[cZ, (1-c)Z]$. If $V > Z$, the new indifference curve Z is closer to the origin O = (0, 0) than the old indifference curve V, and vice versa. Figure 1 is, for convenience, drawn on the presumption that $V > Z$.

The indifference curve $Z[c]$ meets the iso-velocity curve $Z[n]$ at two points, point H and point E defined by

$$E = (Z, Z), \qquad H = \left(\frac{c}{n}Z, \ \frac{1-c}{1-n}Z\right) \tag{20}$$

These two solutions may be the result of substituting (18) into (19). Since it is also likely that $c \neq n$ in general, the trivial solution E would be ruled out and the point H is the 'true' solution. Then the 'true' velocities at H are

$$Z_1 \neq Z_2 \qquad\qquad c \neq n \tag{21}$$

Figure 1 shows the common coordinates for (V_1 , V_2) and (Z_1, Z_2) simultaneously.

The slope, ($-\hat{p}$), of a straight line tangent to the indifference curve Z at point H must equal the ratio of the marginal velocities as follows :

$$-\hat{p} = \left.\frac{dZ_2}{dZ_1}\right|_H \tag{22}$$

Then, at point H the following condition holds from (19) and (22) :

$$\hat{p} = \left.\frac{c}{1-c}\left(\frac{Z_2}{Z_1}\right)^2\right|_H \tag{23}$$

This is the equality of the rate of marginal velocity substitution of Z_1 for Z_2 and the absolute value of the slope $\hat{p}$ at point H. Just as (13), this is also the law of equal marginal velocities per dollar at point $H[\hat{p}]$.

Suppose that the economy was located at point $H[\hat{p}]$ on the old indifference curve $Z[c]$ right before it moved to point $F[p]$ on the new indifference curve $V[a]$. The condition under which point $F[p]$ and point $H[\hat{p}]$ have the identical slope is

$$p = \hat{p} \tag{24}$$

which is the definition of the parallel movement effect in the Fisherian space.

6, The Substitution Effect

The description of the parallel movement effect $p = \hat{p}$ implies that point $H[\hat{p}]$ is the intermediate point that completes the substitution effect and begins the parallel movement effect. Let us choose another point J on the same old indifference curve $Z[c]$, where the slope of the point J is defined by $p_o \neq \hat{p}$, and label the point as $J[p_o]$. The fact that the two points $J[p_o]$ and $H[\hat{p}]$ are located on the same indifference curve $Z[c]$ verifies that the two points share the same average Z and the same weight c . Although their locations on the same indifference curve are different from each other, the coordinate of Z_1 of the point $J[p_o]$ is not the same as that of the point $H[\hat{p}]$ in accordance with the relation $nZ_1 = cZ$ given by (20) so that the money ratio of point $J[p_o]$, denoted by m_o, is different from the money ratio of point $H[\hat{p}]$, denoted by n. Since the two points are not the same, $m_o \neq n$. As a result, we have another iso-velocity line, $Z[m_o]$, connecting point $J[p_o]$ and point E, as follows:

$$Z = m_o Z_1 + (1 - m_o) Z_2 , \qquad m_o \neq n \tag{25}$$

The dotted line, $Z[m_o]$, in Figure 1 is equation (25). Since $m_o \neq n$ and $m_o \neq m$ and $m_o \neq c$, the iso-velocity line (25) is different from the iso-velocity line (18). This implies that $Z[n]$ and $Z[m_o]$ are completely different iso-velocity lines passing through the common indifference curve $Z[c]$ of (19). One of them, the new iso-velocity line $Z[m_o]$ of (25), meets the indifference curve $Z[c]$ at two points; point E and point $J[p_o]$

$$E = (Z,\ Z), \qquad J = \left(\frac{c}{m_o} Z,\ \ \frac{1-c}{1-m_o} Z\right) \tag{26}$$

This is equivalent to the result of substituting (25) into (19). Point $J[p_o]$ is the 'true' solution.

7. Summary

What we have investigated so far may be summarized as follows. The total effect of a change in the slope from p_o to $p\ (\neq p_o)$ in the Fisherian space may be decomposed into the substitution effect from point $J[p_o] = \left(\frac{c}{m_o} Z, \frac{1-c}{1-m_o} Z\right)$ to point $H[\hat{p}] = \left(\frac{c}{n} Z, \frac{1-c}{1-n} Z\right)$ along the same indifference curve $Z[c]$ with $p_o \neq \hat{p}$ and the parallel movement effect from point $H[\hat{p}] = \left(\frac{c}{n} Z, \frac{1-c}{1-n} Z\right)$ on the old indifference curve $Z[c]$ to point $F[p] = \left(\frac{a}{m} V, \frac{1-a}{1-m} V\right)$ on the new indifference curve $V[a]$ with the condition that $\hat{p} = p$.[5] Point $J[p_o]$ may be labeled as the Fisherian initial point, point $H[\hat{p}]$ as the Fisherian intermediate point, and point $F[p]$ as the Fisherian final point.

III. The Exchange Equations in the Marshallian Space

There are two approaches in explaining the quantity theory of money. The Fisherian

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5) Another way of stating the overall movement between the initial point and the final point is to define firstly the parallel movement between two different indifference curves and secondly the substitution effect along the same new indifference curve. This possibility will be discussed in footnote 14.

exchange equation and the Cambridge exchange equation. The previous analysis has been conducted on the basis of the Fisherian equations (6) and (17), and the analysis has been labeled as the Fisherain problem for convenience. The Fisherain space is a two-dimensional space. This section is devoted to the same problem on the basis of the Cambridge (or Marshallian) equations and will be labeled as the Marshallian problem for convenience. The Marshallian space is also a two-dimensional space. This section inquires a question as to how the Marshallian space would reflect the Fisherian decomposition described in the above that the substitution effect moves the economy from the initial point $J[p_o]$ to the intermediate point $H[\hat{p}]$ along the same indifference curve with $p_o \neq \hat{p}$ and the parallel movement effect moves the economy further from the intermediate point $H[\hat{p}]$ to the final point $F[p]$ on the different indifferent curve with the condition $\hat{p}=p$ remaining unchanged.

1. The Iso-transaction Time Line

The velocity V in the Fisherian equation indicates the number of turnovers *per unit of time*. Then, its inverse $\frac{1}{V} = k$ (the Marshallian k) is defined as the time duration of the flows of goods and services money could purchase, for example, the average number of weeks or months income held in the form of money balances. Let us call it transactions time for convenience and its derivative the marginal transactions duration or time. We may use the definition of the variable transformation formula $kV\equiv 1$, $k_1 V_1 \equiv 1$, $k_2 V_2 \equiv 1$ to convert the Fisherian equations (6) into the Marshallian equations (27) as follows :

$$k \equiv \frac{M}{A}, \quad k_1 \equiv \frac{M_1}{A_1}, \quad k_2 \equiv \frac{M_2}{A_2} \qquad (27)\ [=(6)]$$

where (27)=(6) means they are arithmetically equivalent. Substituting (27) into (5) yields

$$k = ak_1 + (1-a)k_2 \qquad (28)\ [=(9)]$$

According to (28), the average transactions time k is the 'arithmetic mean' of the first sector's transaction time k_1 and the second sector's transaction time k_2 with the weight being a and $1-a$, respectively. Equation (28) may be labeled as the iso-time line of k, which is algebraically equivalent to the indifference curve of V in (9) with the help of the variable transformation. Thus, the relationship between the Fisherian space and the Marshallian space is the '*harmonic mean*' and '*arithmetic mean*' relation algebraically and the *curve* and *line* relation geometrically. The iso-time line (28) is labeled as $k[a]$ in Figure 2, which represents the Marshallian space. The iso-time line $k[a]$ in Figure 2 is the *Marshallian image* of the Fisherian velocity indifference curve $V[a]$ in Figure 1.

Here is the place of <Figure 2>

2. The Transaction Time Indifference Curve

Let us substitute the Marshallian equation (27) into (4) to obtain the following rectangular hyperbola :

$$\frac{1}{k} = \frac{m}{k_1} + \frac{1-m}{k_2} \qquad (29)\ [=(7)]$$

Equation (29) is a rectangular hyperbola with the center $O_k = [mk,\ (1-m)k]$ in the Marshallian coordinates of (k_1, k_2), which may be obtained in an alternative way by transforming the iso-velocity line of V in (7) in the Fisherian space of (V_1, V_2). Thus (29)=(7). This is the case that the 'harmonic mean' in the Marshallian space becomes the 'arithmetic mean' in the Fisherian space. The harmonic mean (29) may be called the transactions time indifference curve and denoted by $k[m]$. As expected, the time indifference curve $k[m]$ in Figure 2 is the *Marshallian image* of the Fisherian iso-velocity line $Z[m]$ in Figure 1.

Figure 2 shows that the indifference curve $k[m]$ and the iso-time line $k[a]$ meet at two points, point $F^{\perp}$ and point $E^{\perp}$.

$$E^{\perp} = (k,\ k), \qquad F^{\perp} = \left(\frac{m}{a}k,\ \frac{1-m}{1-a}k\right) \qquad (30)\ [=(10)]$$

One may obtain the same result by substituting the arithmetic mean (28) into the harmonic mean (29). Since we can verify that $E^{\perp}E = I$ and $F^{\perp}F = I$, the Marshallian solution (30) is arithmetically equivalent to the Fisherian solution (10) and the point $F^{\perp}$ is the 'true' solution in the Marshallian space as much as the point F is in the Fisherian space.

3. The Law of Equal Marginal Transactions Time

Let us take a close look at point $F^{\perp}$ on the indifference curve $k[m]$. The law of equal marginal transactions time is given from (29) as follows :

$$q = \frac{m}{1-m}\left(\frac{k_2}{k_1}\right)^2 \Bigg|_{F^{\perp}} \tag{31}$$

where q is the slope of the tangent line to the point $F^{\perp}$. The fact that $1 \neq a+m$ in general ensures that $pq \neq 1$ from (13) and (31). This implies that the law of equal marginal transactions time (31) is independent of the law of equal marginal transactions velocity (13). We denote point $F^{\perp}$ as $F^{\perp}[q]$. The Marshallian final point $F^{\perp}[q]$ in Figure 2 is *the Marshallian image* of the Fisherian final point $F[p]$ in Figure 1.

4. The Parallel Movement Effect

The previous section illustrates the properties of the Marshallian final point $F^{\perp}[q]$ in Figure 2, coupled with its image, the Fisherian final point $F[p]$ in Figure 1. The same manner would be applied to the investigation of the properties of the Marshallian intermediate point $H^{\perp}[\hat{q}]$ in Figure 2 in comparison with its image, the Fisherian intermediate point $H[\hat{p}]$ in Figure 1. With the aid of definitions $ZK \equiv 1$, $Z_1K_1 \equiv 1$, and $Z_2K_2 \equiv 1$, the Fisherian equations (17) may be transformed as follows:

$$K \equiv \frac{N}{C}, \quad K_1 \equiv \frac{N_1}{C_1}, \quad K_2 \equiv \frac{N_2}{C_2} \qquad (32)\ [=(17)]$$

Substituting the Marshallian equations (32) into the definition (16) gives us

$$K = cK_1 + (1-c)K_2 \qquad (33)\ [=(19)]$$

This is another iso-time line. Now substituting the Marshallian equations (32) into the definition (15) yields

$$\frac{1}{K} = \frac{n}{K_1} + \frac{1-n}{K_2} \qquad (34)\ [=(18)]$$

This is another transactions time indifference curve. Figure 2 shows both of the Marshallian coordinates (k_1, k_2) and (K_1, K_2) simultaneously. Since the harmonic mean expressed in (34) is characterized by K and n, it would be denoted by $K[n]$. By the same manner the arithmetic mean (33) may be denoted by $K[c]$. $Z < V$ in Figure 1 means that $K > k$ in Figure 2. Hence, the location of $k[m]$ is closer to the origin $O = (0, 0)$ than the location of $K[n]$.

Substituting the arithmetic mean of (33) into the harmonic mean of (34) gives rise to two solutions.

$$E^{\perp} = (K,\ K), \qquad H^{\perp} = \left(\frac{n}{c}K,\ \frac{1-n}{1-c}K\right) \qquad (35)\ [= (20)\ ]$$

In Figure 2, the indifference curve $K[n]$ meets the iso-time line $K[c]$ at two points, point $H^{\perp}$ and point $E^{\perp}$. One may verify that the Marshallian solution (35) is the image of the Fisherian solution (20) with the aid of $E^{\perp}E = I$ and $H^{\perp}H = I$. Since $n \neq c$, point $H^{\perp}$ in the Marshallian space is as 'true' a point as point H in the Fisherian space.

Let us consider point $H^{\perp}$ on the indifference curve $K[n]$ as the true solution in the Marshallian space of Figure 2, and define $\hat{q}$ as the slope of the tangent line to that point. Then, one may have the law of equal marginal transactions time from (34) as follows:

$$\hat{q} = \frac{n}{1-n}\left(\frac{K_2}{K_1}\right)^2 \Bigg|_{H^{\perp}} \qquad (36)$$

The fact that $1 \neq c + n$ in general ensures that $1 \neq \hat{p}\hat{q}$ from (23) and (36), which implies independence between the law of equal marginal velocity (23) and the law of equal marginal time (36). Using $\hat{q}$ at point $H^{\perp}$, we denote it as $H^{\perp}[\hat{q}]$. The point $H^{\perp}[\hat{q}]$ in Figure 2 is the Marshallian intermediate point as the Marshallian image of the Fisherian intermediate point $H[\hat{p}]$ in Figure 1 formed by the cross of the iso-velocity line $Z[n]$ and the velocity indifference curve $Z[c]$. The parallel movement effect in the Marshallian space may be also defined by the movement from the Marshallian intermediate point $H^{\perp}[\hat{q}]$ to the Marshallian final point $F^{\perp}[q]$ with the condition

$$q = \hat{q} \qquad (37)$$

being held as much as the parallel movement effect in the Fisherian space is defined by the movement from the Fisherian intermediate point $H[\hat{p}]$ to the Fisherian final point $F[p]$ with the condition $p = \hat{p}$ being held in (24).

5. The Substitution Effect

Final attention should be paid to the substitution effect in the Marshallian space for completion. Let us investigate the location of the Marshallian initial point $J^{\perp}[q_o]$, the Marshallian image of the Fisherian initial point $J[p_o]$ expressed in (26). The Fisherian initial point $J[p_o]$ is the product of the intersection of the velocity indifference curve $Z[c]$ in (19) and the iso-velocity line $Z[m_o]$ in (25). This observation guarantees us that the Marshallian initial point $J^{\perp}[q_o]$ must be located on the indifference curve $K[m_o]$ defined by

$$\frac{1}{K} = \frac{m_o}{K_1} + \frac{1-m_o}{K_2} \qquad (38)\ [=(25)]$$

which is transformed from the Fisherian iso-velocity line $Z[m_o]$ defined by (25). The indifference curve $K[m_o]$, defined by (38) and depicted as a dotted curve in Figure 2, is entirely different from the indifference curve $K[n]$ defined by (34) since the center of the indifference curve (38) is $[m_oK, (1-m_o)K]$, while the center of the indifference curve (34) is $[nK, (1-n)K]$. This is due to the fact that the weight n and the average transactions time K creating the unique indifference curve $K[n]$ makes $K[m_o] \neq K[n]$. In other words, the comparison of the indifference curve $K[m_o]$ with the indifference curve $K[n]$ makes it possible that the two curves should meet once at the common point $E^{\perp}$ on the 45 degree line because of the same average transactions time K. However, the overall curvatures are different within the entire range of the curves because of the different weights $m_o \neq n$. As a result, it is impossible that the Marshallian initial point $J^{\perp}[q_o]$ is located on the same indifference curve $K[n]$ on which the Marshallian intermediate point $H^{\perp}[\hat{q}]$ is located. Rather it is located on the different new indifference curve $K[m_o]$. Thus the Marshallian initial point $J^{\perp}[q_o]$ is generated by the intersection of the new indifference curve $K[m_o]$ and the iso-time line $K[c]$. q_o is the slope of the tangent line to the indifference curve $K[m_o]$ at the point $J^{\perp}[q_o]$.

6. Summary

The previous analysis ensures us the fact that the movement from the Marshallian initial point $J^{\perp}[q_o]$ to the Marshallian intermediate point $H^{\perp}[\hat{q}]$ is the exact duplicate of the movement from the Fisherian initial point $J[p_o]$ to the Fisherian intermediate point $H[\hat{p}]$. However, one important difference should be mentioned. The movement from the initial point $J[p_o]$ to the intermediate point $H[\hat{p}]$ along the "same velocity indifference *curve*" $Z[c]$ in the Fisherian space becomes the movement from the initial point $J^{\perp}[q_o]$ to the intermediate point $H^{\perp}[\hat{q}]$ along the "same iso-transaction time *line*" $K[c]$ in the Marshallian space. This result forces us to recognize the fact that it is impossible to define the co-existence of the substitution effect along the same indifference curve simultaneously in the Fisherian space as well as the Marshallian space.

Definition 1 : The Hicks strong substitution effect is the movement from the initial point to the intermediate point *along the same indifference curve* defined by the same parameter (weight) and the same magnitude (average) *coupled with different slopes in both spaces.*

IV. Impossibility of the Hicks Strong Substitution Effect and Strong Parallel Movement Effect

1. Impossibility of the Hicks Strong Substitution Effect

The previous section recreates the Fisherian comparative analysis in terms of the Marshallian space. One interesting point we find is that when the Fisherian initial point $J[p_o]$ and the Fisherian intermediate point $H[\hat{p}]$ are located *with different slopes* on the *same indifference curve* $Z[c]$ in the Fisherian space, the Marshallian initial point $J^{\perp}[q_o]$ is located on the indifferent curve $K[m_o]$ and the Marshallian intermediate point $H^{\perp}[\hat{q}]$ is located on the *different indifference curve* $K[n]$ in the Marshallian space, or the two points $J^{\perp}[q_o]$ and $H^{\perp}[\hat{q}]$ are located *with the same slopes* on the *same iso-transaction time line* $K[c]$ in the Marshallian space. This implies that if the strong substitution effect is defined in the Fisherian space, its image in the Marshallian space cannot be defined, and vice versa.

Theorem 1 (Impossibility of the Hicks Strong Substitution Effect) : The Hicks strong substitution effect described by the movement from the initial point $J[p_o]$ to the intermediate point $H[\hat{p}]$ along the same indifference curve $Z[c]$ characterized by the average Z and the weight c in the Fisherian space cannot be defined by the Hicks strong substitution effect expressed by the movement from the initial point $J^{\perp}[q_o]$ to the intermediate point $H^{\perp}[\hat{q}]$ along the same indifference curve $K[n]$ characterized by the average K and the weight n in the Marshallian space. The reverse is also true. [6]

6) If the strong substitution effect is defined in the Marshallian space, its counterpart in the

Proof : Consider the Fisherian movement between point $R = [Z_1, Z_2]$ and point $R' = [Z_1', Z_2']$ along the same indifference curve $Z[c]$ in the Fisherian space. Consider further the Marshallian movement between point $R^{\perp} = [K_1, K_2]$ and point $(R')^{\perp} = [K_1', K_2']$ along the same indifference curve $K[n]$ in the Marshallian space, which is the counterpart of the Fisherian movement. Our purpose is to prove that $R = R'$ and $R^{\perp} = (R')^{\perp}$. Let us define

$$\frac{Z_2'}{Z_1'} = (\lambda')\left(\frac{Z_2}{Z_1}\right), \qquad \lambda' = \frac{n'}{1-n'}\,\frac{1-n}{n}\,\frac{1-c'}{c'}\,\frac{c}{1-c} \tag{39}$$

First, we investigate the Fisherian space. We know by definition that the transactions weight is $c = c'$ along the same indifference curve in the Fisherian space. As a result, it is true that the following same law of equal marginal velocity at point R and point R' with $c = c'$ is established.

$$\hat{p}' = \frac{c}{1-c}\left(\frac{Z_2'}{Z_1'}\right)^2 \Bigg|_{R'}, \qquad \hat{p} = \frac{c}{1-c}\left(\frac{Z_2}{Z_1}\right)^2 \Bigg|_{R} \tag{40}$$

It is also true that in order for point $R = [Z_1, Z_2]$ to be different from point $R' = [Z_1', Z_2']$ the sectoral money ratio should be different from point to point even though they are located on the same indifference curve in the Fisherian space so that $n' \neq n$ and $c = c'$. As a result, the duplicate of (40) in the Marshallian space with the same condition that $n' \neq n$ and $c = c'$ is

$$\hat{q}' = \frac{n'}{1-n'}\left(\frac{K_2'}{K_1'}\right)^2, \qquad \hat{q} = \frac{n}{1-n}\left(\frac{K_2}{K_1}\right)^2 \tag{41}$$

But these are not located on the same indifference curve $K[n]$. By definition of the same indifference curve, the weight should be $n = n'$ along the same indifference curve $K[n]$ in the Marshallian space. Thus, the money ratio of the two points $R^{\perp}$ and $(R')^{\perp}$ should be $n = n'$. For the two movements along the 'same' indifferent curve in each space to be identical, (41) should become

$$\left(\frac{1}{\lambda'}\right)\hat{q}' = \frac{n}{1-n}\left(\frac{K_2'}{K_1'}\right)^2 \Bigg|_{(R')^{\perp}}, \qquad \hat{q} = \frac{n}{1-n}\left(\frac{K_2}{K_1}\right)^2 \Bigg|_{R^{\perp}} \tag{42}$$

Fisherian space cannot be defined.

with the help of (39). As a result, we have from (39)~(42)

$$\frac{\hat{p}'}{\hat{q}'} = (\lambda')^3 \frac{\hat{p}}{\hat{q}} \quad , \quad \hat{p}' = (\lambda')^2 \hat{p} \quad , \quad \hat{q}' = \left(\frac{1}{\lambda'}\right)\hat{q} \tag{43}$$

Next, turn to the Marshallian space. As said in the above, since $n = n'$ at every point along the same indifference curve, the following same law of equal marginal transactions time is held at the two different points $R^{\perp}$ and $(R')^{\perp}$ on the same indifference curve $K[n]$ with $n = n'$.

$$\hat{q}' = \frac{n}{1-n}\left(\frac{K_2'}{K_1'}\right)^2 \Bigg|_{(R')^{\perp}} \quad , \quad \hat{q} = \frac{n}{1-n}\left(\frac{K_2}{K_1}\right)^2 \Bigg|_{R^{\perp}} \tag{44}$$

In order for point $R^{\perp} = [K_1, K_2]$ to be different from point $(R')^{\perp} = [K_1', K_2']$ the sectoral transaction ratio should be different from point to point even though they are located on the same indifference curve in the Marshallian space so that $c' \neq c$ and $n = n'$. Thus the counterpart of (44) in the Fisherian space with the same condition that $c' \neq c$ and $n = n'$ is

$$\hat{p}' = \frac{c'}{1-c'}\left(\frac{Z_2'}{Z_1'}\right)^2 \quad , \quad \hat{p} = \frac{c}{1-c}\left(\frac{Z_2}{Z_1}\right)^2 \tag{45}$$

But these are not located on the same indifference curve $Z[c]$. By definition the weight is $c = c'$ along the same indifference curve $Z[c]$ in the Marshallian space. Thus, the transaction ratio of the two points R and R' should be $c = c'$. For the two movements along the same indifferent curve in each space to be identical, (45) should become

$$(\lambda')(\hat{p}') = \frac{c}{1-c}\left(\frac{Z_2'}{Z_1'}\right)^2 \Bigg|_{R'} \quad , \quad \hat{p} = \frac{c}{1-c}\left(\frac{Z_2}{Z_1}\right)^2 \Bigg|_{R} \tag{46}$$

with the help of (39). As a result, we have from (44)~(45)

$$\frac{\hat{p}'}{\hat{q}'} = (\lambda')^3 \frac{\hat{p}}{\hat{q}} \quad , \quad \hat{p}' = (\lambda') \hat{p} \quad , \quad \hat{q}' = \left(\frac{1}{\lambda'}\right)^2 \hat{q} \tag{47}$$

Therefore, we have from (43) and (47)

$$\lambda' = 1 \quad , \quad \hat{p}' = \hat{p} \quad , \quad \hat{q}' = \hat{q} \tag{48}$$

From (39)~(41) and (48), $R = R'$ and $R^{\perp} = (R')^{\perp}$. Q.E.D.

2. The Hicks Weak Substitution Effect

The movement from the initial point $J[p_o]$ to the intermediate point $H[\hat{p}]$ along *the same indifference curve* $Z[c]$ in the Fisherian space must correspond to the movement from the initial point $J^{\perp}[q_o]$ to the intermediate point $H^{\perp}[\hat{q}]$ along *the same iso-transactions time line* $K[c]$ in the Marshallian space. Thus, we have

Definition 2 : The Hicks weak substitution effect is the movement from the initial point to the intermediate point *along the same indifference curve* defined by the same parameter (weight) and the same magnitude (average) *coupled with different slopes at least in one space.*

3. Impossibility of the Hicks Strong Parallel Movement Effect

Impossibility of the simultaneous achievement of the strong substitution effects in both spaces leads one to a situation where the simultaneous achievement of the strong parallel movement effects in both spaces is impossible. Here the word "strong" is meant by referring the same slope of the two tangent lines to the two different indifference curve, respectively.[7]

Definition 3 : The Hicks strong parallel movement effect is the movement from the intermediate point on the old indifference curve to the final point on the new indifference curve, at which the tangent slope is the same as the tangent slope at the former in *both spaces*.

Definition 4 : The Hicks weak parallel movement effect [8] is the movement from the

7) The Hicks parallel movement effect is the movement between the two different indifference curves with the condition that the slope of the tangent line to a point on each indifference curve remains the same. Let us call this movement with a strong condition the strong parallel movement effect. Since there exists a unique slope tangent to a point on a indifferent curve, the tangent point on each indifference curve with the same slope is unique. Thus, the point on an indifference curve which is associated with the strong parallel movement effect has one and only one corresponding point on the other indifference curve.

8) The strong parallel movement is meant by the perfect parallel movement in both spaces, while the weak parallel movement is meant by the imperfect parallel movement at least in one space.

intermediate point on the old indifference curve to the final point on the new indifference curve at which the tangent slope is (un)parallel to the slope at the former *at least in one space*.

Before we proceed further, we can provide an intuitive explanation of the impossibility of the Hick strong parallel movement effect. When the indifference *curve* moves from $Z[c]$ to $V[a]$ in the Fisherian space, the corresponding iso-time *line* moves from $K[c]$ to $k[a]$ in the Marshallian space. Even if the Fisherian parallel movement effect accompanies the same slope by definition, the corresponding Marshallian movement cannot carry the same slope because the slope of the old iso-time line $K[c]$ is $-\frac{c}{1-c}$, while the slope of the new iso-time line $k[a]$ is $-\frac{a}{1-a}$. The two slopes are not identical since $a \neq c$, which implies that in the Marshallian space it is impossible to move between the two points with the identical slope being maintained. This intuitive explanation will be thoroughly reexamined by the comparison of the slopes p and q with the slopes $\hat{p}$ and $\hat{q}$, respectively.

Theorem 2 (Impossibility of the Hicks Strong Parallel Movement Effect) : It is impossible to meet simultaneously the condition $\hat{p} = p$ and $\hat{q} = q$, which satisfies the movement from the intermediate points H and $H^{\perp}$ on the old indifference curves where $Z_1 \neq Z_2$ ($K_1 \neq K_2$) to the final points F and $F^{\perp}$ where $V_1 \neq V_2$ ($k_1 \neq k_2$) on the new indifference curves when the average transactions velocity (time) increases (decreases) from Z (K) to V (k) with $a \neq c$ ($m \neq n$) in general.

Proof : We select independent equations among the equations introduced in the above which describe the movement of $H[\hat{p}] \rightarrow F[p]$ with $\hat{p} = p$ in the Fisherian space and the movement of $H^{\perp}[\hat{q}] \rightarrow F^{\perp}[q]$ with $\hat{q} = q$ in the Marshallian space. The simultaneous equations system for the strong parallel movement is given as follows.

<Simultaneous equations system I for the Strong Parallel Movement>

$V[a]$	$\frac{1}{V} = \frac{a}{V_1} + \frac{1-a}{V_2}$	(9)	$V[m]$	$V = mV_1 + (1-m)V_2$	(7)
$F[p]$	$\frac{a}{1-a}\left(\frac{V_2}{V_1}\right)^2 \Big\vert_F = p$	(13)	$F^{\perp}[q]$	$\frac{m}{1-m}\left(\frac{V_1}{V_2}\right)^2 \Big\vert_{F^{\perp}} = q$	(31)
$Z[c]$	$\frac{1}{Z} = \frac{c}{Z_1} + \frac{1-c}{Z_2}$	(19)	$Z[n]$	$Z = nZ_1 + (1-n)Z_2$	(18)
$H[\hat{p}]$	$\frac{c}{1-c}\left(\frac{Z_2}{Z_1}\right)^2 \Big\vert_H = \hat{p}$	(23)	$H^{\perp}[\hat{q}]$	$\frac{n}{1-n}\left(\frac{Z_1}{Z_2}\right)^2 \Big\vert_{H^{\perp}} = \hat{q}$	(36)

The definitions of the variable transformation formula $k_1 V_1 \equiv 1$, $k_2 V_2 \equiv 1$, $K_1 Z_1 \equiv 1$, $K_2 Z_2 \equiv 1$ have been already employed in (31) and (36). We add to simultaneous equations system I the two definitions of the Hick strong parallel movement effect:

$$\hat{p} = p\ , \qquad \hat{q} = q \tag{49}$$

The system consists of 10 independent equations with 14 unknowns (V_1, V_2, Z_1, Z_2, m, n, p, $\hat{p}$, q, $\hat{q}$, a, c, V, Z). The solution may be expressed as a function of 4 variables among them. Thus, we ends up with two solutions as follows :

$$V_1 \neq V_2 \quad \text{with} \quad Z_1 = Z_2 \qquad (50)$$ [9]

or

$$V_1 = V_2, \quad \text{with} \quad Z_1 \neq Z_2 \qquad (51).$$ [10]

The result says that the solution satisfying the condition $\hat{p} = p$ with $\hat{q} = q$ contains the trivial solution, which contradicts the general presumption that the trivial solution is ruled out. Q.E.D.

V. Possibility of the Hicks Weak Parallel Movement Effect

The impossibility theorem of the Hicks strong parallel movement effect in both spaces forces us to search alternative weak conditions with which the Fisherian movement from the intermediate point $H[\hat{p}]$ to the final point $F[p]$ and the Marshallian movement from the intermediate point $H^{\perp}[\hat{q}]$ to the final point $F^{\perp}[q]$ comply. The weak conditions should be one of the three cases ; (i) $\hat{p} = p$ with $\hat{q} \neq q$, (ii) $\hat{p} \neq p$ with $\hat{q} = q$, (iii) $\hat{p} \neq p$ with $\hat{q} \neq q$. But the general case that includes all of the three cases is $\hat{p} \neq p$ with $\hat{q} \neq q$ such that the tangent slope $\hat{p}$ of the intermediate point $H[\hat{p}]$ is *uniquely* associated with the tangent slope p of the final point $F[p]$ on the basis of the one-to-one correspondence in the Fisherian space, and at the same

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9) $V_1 = a\left[1 + \left(\frac{1-a}{a}\frac{1}{p}\right)^{\frac{1}{2}}\right]V$ and $V_2 = (1-a)\left[1 + \left(\frac{a}{1-a}p\right)^{\frac{1}{2}}\right]V$.

10) $Z_1 = c\left[1 + \left(\frac{1-c}{c}\frac{1}{\hat{p}}\right)^{\frac{1}{2}}\right]Z$ and $Z_2 = (1-c)\left[1 + \left(\frac{c}{1-c}\hat{p}\right)^{\frac{1}{2}}\right]Z$.

time the tangent slope $\hat{q}$ of the intermediate point $H^{\perp}[\hat{q}]$ is *uniquely* associated with the tangent slope q of the final point $F^{\perp}[q]$ on the basis of the one-to-one correspondence in the Marshallian space. This is the concept of the Hicks weak parallel movement effect and the weak here means the imperfect while the strong means the perfect.

1. The Hicks Weak Parallel Movement Effect

To identify the Hicks weak parallel movement effect with the condition that $a \neq c$, $m \neq n$, and $V \neq Z$, we take the first step by defining the following two real numbers :

$$\Delta = \frac{(1-a)V-(1-c)Z}{aV-cZ} > 0, \qquad \Omega = \frac{(1-m)k-(1-n)K}{mk-nK} > 0 \tag{52}$$

The first definition stands for the slope of the moving distance of the centers of the two indifference curves in the Fisherian space, $O_Z O_V$, in Figure 1. The second definition represents the slope of the moving distance of the centers of the two indifference curves in the Marshallian space, $O_K O_k$, in Figure 2. We call (52) *the centerline slope*. They are positive because indifference curves cannot intersect. [The proof is omitted].

Theorem 3 (Possibility of the Hicks weak parallel movement effect) : The Hicks weak parallel movement effect is expressed by a *unique* relationship between p and $\hat{p}$ in the Fisherian space, and between q and $\hat{q}$ in the Marshallian space. First, the Fisherian movement from the intermediate point $H[\hat{p}]$ on the indifference curve $Z[c]$ to the final point $F[p]$ on the indifference curve $V[a]$ in the Fisherian space is *uniquely* defined by

$$p = \frac{1}{\hat{p}} \frac{a}{1-a} \frac{c}{1-c} \Delta^2 \tag{53}$$

Second, the Marshallian movement from the intermediate point $H^{\perp}[\hat{q}]$ on the indifference curve $K[n]$ to the final point $F^{\perp}[q]$ on the indifference curve $k[m]$ in the Marshallian space is *uniquely* defined by

$$q = \frac{1}{\hat{q}} \frac{m}{1-m} \frac{n}{1-n} \Omega^2 \tag{54}$$

Proof [11] : In the real number space, there exist two real numbers $\hat{\Delta}$ =

$$\frac{\left(\frac{k}{1-a}\right)-\left(\frac{K}{1-c}\right)}{\left(\frac{k}{a}\right)-\left(\frac{K}{c}\right)}$$ and $\hat{\Omega} = \dfrac{\left(\frac{V}{1-m}\right)-\left(\frac{Z}{1-n}\right)}{\left(\frac{V}{m}\right)-\left(\frac{Z}{n}\right)}$ such that

$$\varDelta^2 = \left(\frac{1-a}{a}\,\frac{1-c}{c}\right)^2 \hat{\varDelta}^2, \qquad \Omega^2 = \left(\frac{1-m}{m}\,\frac{1-n}{n}\right)^2 \hat{\Omega}^2 \tag{55}$$

with the aid of (52). In the same real number space, there exist two real numbers $\frac{V_2'}{V_1'}$ and $\frac{k_2''}{k_1''}$ such that

$$\varDelta^2 = \left(\frac{V_2'}{V_1'}\right)^2 \left(\frac{Z_2}{Z_1}\right)^2, \qquad \Omega^2 = \left(\frac{k_2''}{k_1''}\right)^2 \left(\frac{K_2}{K_1}\right)^2 \tag{56}$$

by substituting (13), (23), (31), and (36) into (55), where

$$\frac{V_2'}{V_1'} = (\mu')\frac{V_2}{V_1}, \qquad \frac{k_2''}{k_1''} = \left(\frac{1}{\mu''}\right)\frac{k_2}{k_1} \tag{57}$$

$$\mu' = \frac{V_2}{V_1}\,\frac{Z_2}{Z_1}\,\frac{\hat{\varDelta}}{p\hat{p}}, \qquad \frac{1}{\mu''} = \frac{k_2}{k_1}\,\frac{K_2}{K_1}\,\frac{\hat{\Omega}}{q\hat{q}} \tag{58}$$

Rearranging (55)~(58) yields

$$p\hat{p} = \frac{a}{1-a}\,\frac{c}{1-c}\left(\frac{1}{\mu'}\,\frac{V_2'}{V_1'}\,\frac{Z_2}{Z_1}\right)^2, \quad q\hat{q} = \frac{m}{1-m}\,\frac{n}{1-n}\left(\mu''\,\frac{k_2''}{k_1''}\,\frac{K_2}{K_1}\right)^2 \tag{59}$$

The first equation of (57) indicates the Fisherian relationship between point $S=[V_1, V_2]$ and point $S'=[V_1', V_2']$ on the same indifference curve in the Fisherian space. The counterpart of this Fisherian relationship described in the Marshallian space is the Marshallian relationship between point $S^{\perp}=[k_1, k_2]$ and point $(S')^{\perp}=[k_1', k_2']$. The two Marshallian points should be located on the same indifference curve so that μ' = 1 in accordance with theorem 1. The same can be said about the second equation of (57), which tells us the relationship between point $S^{\perp}=[k_1, k_2]$ and point $(S'')^{\perp}=[k_1'', k_2'']$ on the same indifference curve. The counterpart of this Marshallian relationship described in the Fisherian space is the

11) The appendix at the end of this paper provides an alternative proof, which is extremely long but verifies the same result. The appendix may be cut off after verification due to its excessive length.

Fisherian relationship between point $S=[V_1, V_2]$ and point $S''=[V_1'', V_2'']$. The two Fisherian points should be located on the same indifference curve so that $\mu''=1$ in accordance with theorem 1. We use these results in (56)~(57) and (59) to get

$$p\hat{p} = \frac{a}{1-a}\frac{c}{1-c}\Delta^2 , \qquad q\hat{q} = \frac{m}{1-m}\frac{n}{1-n}\Omega^2 \tag{60}$$

This is (53)~(54). QED. [The proof for uniqueness is omitted].

2. The Copula between the Fisherian and Marshallian Space

The definition of Hicks weak parallel movement effect (60) represents the general case, $\hat{p}\neq p$ with $\hat{q}\neq q$, of the Hicks weak parallel movement effect, which includes the one special case that $\hat{p}=p$ with $\hat{q}\neq q$ and the other special case that $\hat{p}\neq p$ with $\hat{q}=q$. Another way of stating (60) is simply

$$\Delta\Omega = 1 \tag{61}$$

with the help of (13), (23), (31), and (36). This result is an additional example of the variable transformation formula like $Vk=1$, $V_1k_1=1$, $V_2k_2=1$ between the Fisherian space and the Marshallian space.12) The distinctive feature of (61) is the transformation of the *centerline slope* of one space to another, while $Vk=1$, $V_1k_1=1$, $V_2k_2=1$ are the transformation of *a point* between the two spaces. Thus, *the discovery of (61) may confirms indirectly that the weak parallel movement effect exists in the form of (60).* It is important to notice that (61) still holds even when the one special case that $\hat{p}=p$ with $\hat{q}\neq q$ and the other special case that $\hat{p}\neq p$ with $\hat{q}=q$ are applied to the general case of (60).

VI. The Decomposition of the Transactions Velocity of Money

1. Simultaneous Equations System II for the Weak Parallel Movement

Simultaneous equations system I replaces the definitions of the Hicks strong parallel movement effect (24) and (37) by the definitions of the Hicks weak parallel movement effect (53) and (54) with the rest of the 8 independent equations remaining the same to become simultaneous equations system II for the weak parallel movement. That is,

12) An alternative proof of the expression (61) is given by (A-28) in the appendix at the end of this paper.

<Simultaneous Equations System II for the Weak Parallel Movement>

$$\frac{1}{V} = \frac{a}{V_1} + \frac{1-a}{V_2} \qquad (9) \qquad\qquad V = mV_1 + (1-m)\,V_2 \qquad (7)$$

$$\frac{a}{1-a}\left(\frac{V_2}{V_1}\right)^2 \Bigg|_F = p \qquad (13) \qquad\qquad \frac{m}{1-m}\left(\frac{V_1}{V_2}\right)^2 \Bigg|_{F^\perp} = q \qquad (31)$$

$$\frac{1}{Z} = \frac{c}{Z_1^S} + \frac{1-c}{Z_2^S} \qquad (19) \qquad\qquad Z = nZ_1^S + (1-n)Z_2^S \qquad (18)$$

$$\frac{c}{1-c}\left(\frac{Z_2^S}{Z_1^S}\right)^2 \Bigg|_H = \hat{p} \qquad (23) \qquad\qquad \frac{n}{1-n}\left(\frac{Z_1^S}{Z_2^S}\right)^2 \Bigg|_{H^\perp} = \hat{q} \qquad (36)$$

$$p\hat{p} = \frac{a}{1-a}\,\frac{c}{1-c}\,\Delta^2 \qquad (53) \qquad\qquad q\hat{q} = \frac{m}{1-m}\,\frac{n}{1-n}\,\Omega^2 \qquad (54)$$

Notice that equations (53) and (54) take the place of equations (24) and (37). Notice further that the superscript S in (19), (18), (23), and (36) indicates the intermediate situation, which terminates the substitution movement and starts the parallel movement.

2. Solution

Simultaneous equations system II also consists of 10 independent equations with 14 variables (V_1, V_2, Z_1^S, Z_2^S, m, n, p, $\hat{p}$, q, $\hat{q}$, a, c, V, Z). Although it is true that all the variables are determined in the general equilibrium system, some of them are not observable in practice. It is fortunate that 4 variables (a, c, V, Z) out of 14 variables in the system are observable information making the number of independent equations equal the number of the unknowns. This observation enables us to solve the system. In practice, the ten equations can be reduced to two equations which contains two unknowns m and n:

$$\Delta\Omega = 1 \qquad (61)$$

$$\frac{m}{1-m}\,\frac{n}{1-m} = \frac{a}{1-a}\,\frac{c}{1-c}\,\Delta \qquad (62)$$

(61) is derived from equations (13), (23), (31), (36) and (53)~(54). The rest of the equations gives us (62). By substituting the two equations (61) and (62) into each other with respect to n , we come to the following quadratic equation for the solution

$$Bm^2 + 2\Gamma m + \Phi = 0, \qquad (63)$$

where B, Γ and Φ are functions of 4 knowns (a, c, V, Z).[13] Since it is easily verified without presentation of the proof that $\Gamma^2 - B\Phi > 0$, the quadratic equation (63) has two real solutions. One of them satisfies the condition (8) such that $1 > m > 0$ [The proof is omitted]. Thus, the 'true' solution of (63), the sectoral money ratio at the final point $F[p]$, is given by

$$m = \left.\frac{M_1}{M}\right|_{F(p)} = \frac{\left(\frac{a}{1-a}\frac{c}{1-c}\Delta\right)^{\frac{1}{2}}}{1+\left(\frac{a}{1-a}\frac{c}{1-c}\Delta\right)^{\frac{1}{2}}} - G \tag{64}$$

where

$$G = \frac{\left(\frac{a}{1-a}\frac{c}{1-c}\Delta\right)^{\frac{1}{2}}\left[\,(1-D^{\frac{1}{2}})+\left(D-\frac{V}{Z}\right)^{\frac{1}{2}}\,\right]}{1-\frac{a}{1-a}\frac{c}{1-c}\Delta} \tag{65}$$

$$D = \frac{1-a}{a}\frac{1-c}{c}\frac{1}{\Delta}\left[\frac{1}{2}\frac{\Delta}{1+\Delta}\left(\frac{V}{Z}-1\right)\left(1+\frac{a}{1-a}\frac{c}{1-c}\right)\right]^2+\frac{V}{Z} \tag{66}$$

The relationship between the money ratio m and the parameter (a, c, V, Z) are not known a pri-o-ri.

Next, substituting (64) into (7) and using (9), we have the sectoral velocities at the final point $F[p]$ as follows:

$$\left.\frac{V_1}{V}\right|_{F(p)} = \frac{a\left[1+\left(\frac{a}{1-a}\frac{c}{1-c}\Delta\right)^{\frac{1}{2}}\right]}{(1-G)\left[\frac{a}{1-a}\frac{c}{1-c}\Delta\right]^{\frac{1}{2}} - G} \tag{67}$$

$$\left.\frac{V_2}{V}\right|_{F(p)} = \frac{(1-a)\left[\,1+\left(\frac{a}{1-a}\frac{c}{1-c}\Delta\right)^{\frac{1}{2}}\right]}{1+G\left[1+\left(\frac{a}{1-a}\frac{c}{1-c}\Delta\right)^{\frac{1}{2}}\right]} \tag{68}$$

It is verified that $V_1 \neq V_2$ at the final point $F[p]$ and $V_1 > 0$ and $V_2 > 0$. [The proof is omitted]. The relationship between the sectoral velocity (V_1, V_2) and the parameter (a, c, V, Z) are not known a pri-o-ri. Finally, if we substitute (67)~(68) into system II, $Z_1^S \neq Z_2^S$ at the intermediate point H. [The proof is omitted]. The form of (64) and (67)~(68) is not simple by the looks but simple with numbers in practice. If it is not simple, the calculation for the sectoral velocity formula is tedious and time consuming to do by hand. To be practical use in practice, these must be calculated

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13) $B=\left(1-\frac{a}{1-a}\frac{c}{1-c}\Delta\right)(1+\Delta)$, $\Phi=-\left(\frac{V}{Z}+\Delta\right)\frac{a}{1-a}\frac{c}{1-c}\Delta$,

$\Gamma=\frac{1}{2}\left[\frac{a}{1-a}\frac{c}{1-c}\left(\frac{V}{Z}+\Delta\right)-\left(1-\frac{a}{1-a}\frac{c}{1-c}\Delta\right)+\left(\frac{V}{Z}+\frac{a}{1-a}\frac{c}{1-c}\right)\right]\Delta$

quickly and repeatedly. Accordingly, it is essential that the practitioners have computer facilities available.[14)]

VII. Conclusions

This paper discusses the basic method behind the bifurcation process of decomposition of the economy-wide velocity. This process can continue to any sub-level of transactions in a straightforward manner. The importances of finding disaggregated monies and their velocities should not be overlooked.

In practice, for example, we could think of an immediate application to a country's annual series of real and financial transactions. Or one may find the decomposition useful in analyzing the effect of the newly born euro money on each member country's velocity after the European monetary union is established. Identifying regional or industrial monies and their velocities would assist the monetary authorities to allocate the limited amount of available credit more effectively. From the macroeconomic point of

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14) As we have witnessed thus far, this system can solve the final point $F[p]$, but cannot solve the initial point $J[p_o]$. [The proof is omitted]. This is due to the decomposition of the overall movement into the substitution effect from point $J[p_o]$ to point $H[\hat{p}]$ along the same old indifference curve $Z[c]$ and the parallel movement effect from point $H[\hat{p}]$ in the old indifference curve $Z[c]$ to point $F[p]$ in the new indifference curve $V[a]$. The second way of expressing the same overall movement is to define firstly the parallel movement effect from point $J[p_o]$ in the old indifference curve $Z[c]$ to point $X[\hat{\hat{p}}]$ in the new indifference curve $V[a]$, and define secondly the substitution effect from point $X[\hat{\hat{p}}]$ to point $F[p]$ in the new indifference curve $V[a]$. This system can solve the initial point $J[p_o]$, but cannot solve the final point $F[p]$. However, the solution of the initial point $J[p_o]$ is inappropriate and hence the second way is ruled out because the solution is a function of not only c and Z of the initial point $J[p_o]$ but also a and V of the final point $F[p]$ which are not known at the initial point $J[p_o]$. That is, the solution of the initial point $J[p_o]$ in the second way of decomposition is forwarding. This implies that the solution of the final point $F[p]$ in the second way is forwarding as well. This is the reason the final point $F[p]$ cannot be solved in the second way. On the contrary, the solution of the final point $F[p]$ in the first way of decomposition is backward and a function of its contemporary weight a and magnitude V, plus its previous weight c and magnitude Z all of which are known at the final point, as we saw in equation (67) and (68). By the same token, the solution of the initial point $J[p_o]$ in the second way is determined by not only its contemporary weight c and magnitude Z but also its previous weight and magnitude, which are known but not included in the simultaneous equation system in the above. This is the reason we take the backward solution.

view, the knowledge of financial and real velocities over time would be critically important in analyzing the episode like " the great velocity decline." The velocity series for real income process would provide clues on the short-run and long-run effects of monetary policy.

In theory, the decomposition of the transaction velocity of money presented in this paper would provide the general method for the reverse case that any aggregate measure is disaggregated into sub-level components with limited information. One could apply the same method to the capital asset pricing model, among others, to calculate the beta, a measure of risk and an weight given to the linear combination of different rates of return. In general, the possibilities of potential application to any linear combination seem unlimited.

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Figure 1. The Decomposition of the Transactions Velocity in the Fisherian Space

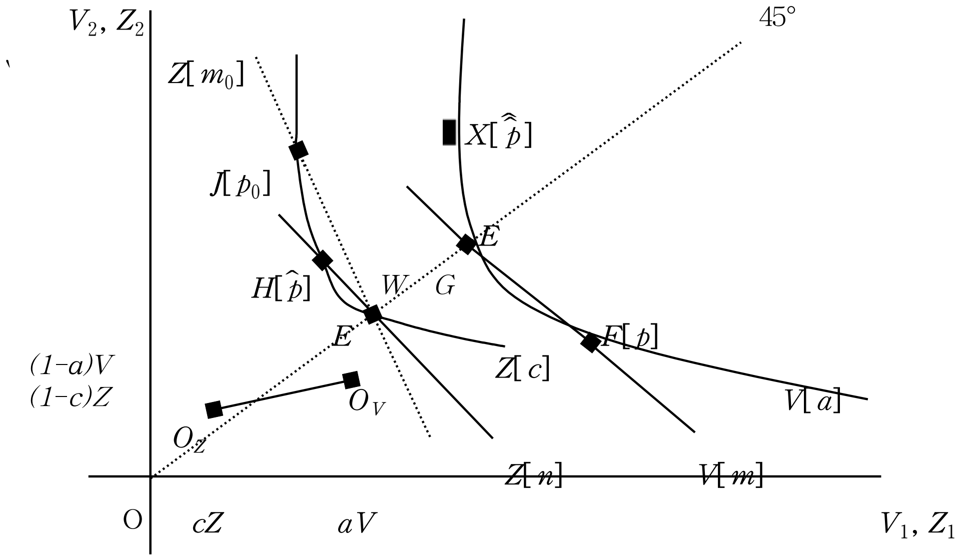
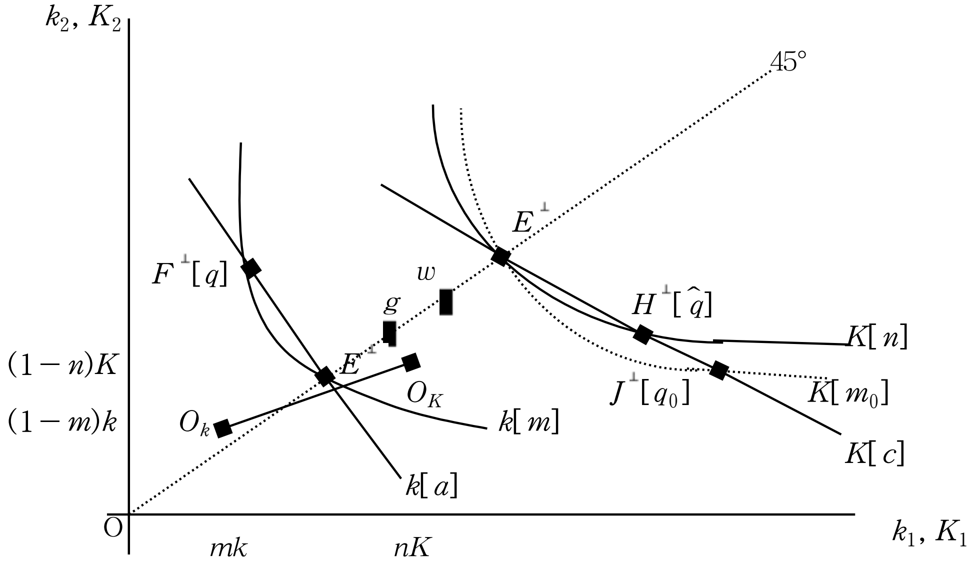


Figure 2. The Decomposition of the Transactions Time in the Marshallian Space



Appendix on Alternative Proof of Theorem 3 [Equation (53) and (54)]

This appendix may be cut off due to its excessive length after the verification of the same result as the one given by the proof in the main text.

Step 1. Our primary concern is to establish the relationship between the slope p of point $F[p]$ and the slope $\hat{p}$ of point $H[\hat{p}]$, which is described by equation (53) in the Fisherian space. This relationship, as the exact copy of the Marshallian space, should be reflected by equation (54) between the slope q of point $F^{\perp}[q]$ and the slope $\hat{q}$ of point $H^{\perp}[\hat{q}]$. In the Fisherian space of <Figure 1>, we take point $G=(G, G)$ in the place where the straight line FH, connecting point $F[p]$ and point $H[\hat{p}]$, meets the 45 degree line. Then, the slope of line FH is defined by

$$\text{slope of } FH = \frac{V_2-Z_2}{V_1-Z_1} = \frac{V_2-G}{V_1-G} \quad \text{or} \quad \frac{V_2-Z_2}{V_1-Z_1} = \frac{Z_2-G}{Z_1-G} \tag{A-1}$$

At the same time in the Marshallian space of <Figure 2>, we take point $w=(w, w)$ in the place where the straight line $F^{\perp}H^{\perp}$, connecting point $F^{\perp}[q]$ and point $H^{\perp}[\hat{q}]$, meets the 45 degree line. Then, the slope of line $F^{\perp}H^{\perp}$ is defined by

$$\text{slope of } F^{\perp}H^{\perp} = \frac{k_2-K_2}{k_1-K_1} = \frac{k_2-w}{k_1-w} \quad \text{or} \quad \frac{k_2-K_2}{k_1-K_1} = \frac{K_2-w}{K_1-w} \tag{A-2}$$

Step 2. Theorem 2 implies implicitly that it is possible to meet simultaneously the double condition $p=\hat{p}$ and $q=\hat{q}$ in both spaces if one of the two solutions is allowed to be trivial on the 45 degree line, which is solution (50) or (51). To achieve this trivial solution, we add in imagination an *instrumental indifference curve* $G[c_G]$ which cuts through the 45 degree line at point $G=(G,G)$ in the Fisherian space of <Figure 1>:

$$\frac{1}{G} = \frac{c_G}{G_1} + \frac{1-c_G}{G_2} \tag{A-3}$$

The slope of the tangent line to at a point of this instrumental indifference curve $G[c_G]$ is defined as $\hat{p}_G$, Then, we have from the law of equal marginal velocities

$$\hat{p}_G = \frac{c_G}{1-c_G}\left(\frac{G_2}{G_1}\right)^2 \tag{A-4}$$

In the Marshallian space of <Figure 2>, we also add in imagination the counterpart indifference curve $g[n_g]$ to this Fisherian instrumental indifference curve $G[c_G]$;

$$\frac{1}{g} = \frac{n_g}{g_1} + \frac{1-n_g}{g_2} \tag{A-5}$$

where $gG=1$, $g_1G_1=1$, $g_2G_2=1$. The slope of the tangent line to a point of this indifference curve $g[n_g]$ is defined as $\hat{q}_g$. Then, we have the law of equal marginal transactions time

$$\hat{q}_g = \frac{n_g}{1-n_g}\left(\frac{g_2}{g_1}\right)^2 \tag{A-6}$$

The above functions form the simultaneous equations system for the relationship between a point of the indifference curve $G[c_G]$ and point $H[\hat{p}]=(Z_1, Z_2)$ of the indifference curve $Z[c]$ in the Fisherian space and the relationship between a point of the indifference curve $g[n_g]$ and point $H^{\perp}[\hat{q}]$ of the indifference curve $K[n]$ in the Marshallian space as follows:

$$\frac{1}{Z} = \frac{c}{Z_1} + \frac{1-c}{Z_2} \qquad\qquad Z = nZ_1 + (1-n)Z_2$$

$$\hat{p} = \frac{c}{1-c}\left(\frac{Z_2}{Z_1}\right)^2 \qquad\qquad \hat{q} = \frac{n}{1-n}\left(\frac{Z_1}{Z_2}\right)^2$$

$$\frac{1}{G} = \frac{c_G}{G_1} + \frac{1-c_G}{G_2} \qquad\qquad G = n_g G_1 + (1-n_g)G_2$$

$$\hat{p}_G = \frac{c_G}{1-c_G}\left(\frac{G_2}{G_1}\right)^2 \qquad\qquad \hat{q}_g = \frac{n_g}{1-n_g}\left(\frac{G_1}{G_2}\right)^2$$

To investigate the co-presence of the strong parallel movement effects in both spaces, we add the following parallel movement conditions

$$\hat{p} = \hat{p}_G\ , \qquad \hat{q} = \hat{q}_g \tag{A-7}$$

These conditions are equivalent to the conditions (49) in the main text. The solution to this system, according to Theorem 2, is the following :

$$G_1 = G_2 = G \qquad\qquad g_1 = g_2 = g \tag{A-8}$$

$$Z_1 = c\left[1+\left(\frac{1-c}{c}\frac{1}{\hat{p}}\right)^{\frac{1}{2}}\right]Z \qquad K_1 = n\left[1+\left(\frac{1-n}{n}\frac{1}{\hat{q}}\right)^{\frac{1}{2}}\right]K \tag{A-9}$$

$$Z_2 = (1-c)\left[1+\left(\frac{c}{1-c}\hat{p}\right)^{\frac{1}{2}}\right]Z \qquad K_2 = (1-n)\left[1+\left(\frac{n}{1-n}\hat{q}\right)^{\frac{1}{2}}\right]K \tag{A-10}$$

The solution is the specific form of the solution (51) in the main text [see footnote 10].

Step 3. The same step is taken for the indifference curve $V[a]$ in the Fisherian space of <Figure 1>. We take another point $W=(W, W)$ between lower point E and point G on the 45 degree line where another instrumental indifference curve $W[a_W]$ passes through ;

$$\frac{1}{W} = \frac{a_W}{W_1} + \frac{1-a_W}{W_2} \tag{A-11}$$

The slope of the tangent line to a point of this indifference curve $W[a_W]$ is defined as p_W. This leads us to the law of equal marginal velocities

$$p_W = \frac{a_W}{1-a_W}\left(\frac{W_2}{W_1}\right)^2 \tag{A-12}$$

Now, we further imagine the counterpart indifference curve $w[m_w]$ in the Marshallian space to the Fisherian indifference curve $W[a_W]$ as follows:

$$\frac{1}{w} = \frac{m_w}{w_1} + \frac{1-m_w}{w_2} \tag{A-13}$$

where $wW=1$, $w_1W_1=1$, $w_2W_2=1$. The slope of the tangent line to a point of this indifference curve $w[m_w]$ is defined as q_w. This leads us to the law of equal marginal transactions time

$$q_w = \frac{m_w}{1-m_w}\left(\frac{w_1}{w_2}\right)^2 \tag{A-14}$$

The above equations form the simultaneous equations system for the relationship between point $F[p]=(V_1, V_2)$ of the indifference curve $V[a]$ and a point of the indifference curve $W[a_W]$ in the Fisherian space and the relationship between point $F^{-1}[q]$ of the indifference curve $V[m]$ and a point of the indifference curve $W[m_w]$

in the Marshallian space as follows:

$$\frac{1}{V}=\frac{a}{V_1}+\frac{1-a}{V_2} \qquad V=m\,V_1+(1-m)V_2$$

$$p=\frac{a}{1-a}\left(\frac{V_2}{V_1}\right)^2 \qquad q=\frac{m}{1-m}\left(\frac{V_1}{V_2}\right)^2$$

$$\frac{1}{W}=\frac{a_W}{W_1}+\frac{1-a_W}{W_2} \qquad W=m_w W_1+(1-m_w)W_2$$

$$p_W=\frac{a_W}{1-a_W}\left(\frac{W_2}{W_1}\right)^2 \qquad q_w=\frac{m_w}{1-m_w}\left(\frac{W_1}{W_2}\right)^2$$

To investigate the co-existence of the strong parallel movement effects in both spaces, we add the following conditions

$$p=p_W, \qquad q=q_w \tag{A-15}$$

These conditions are also equivalent to the conditions (49) in the main text. The solution to this system, according Theorem 2, is

$$W_1=W_2=W \qquad w_1=w_2=w \tag{A-16}$$

$$V_1=\ a\left[1+\left(\frac{1-a}{a}\frac{1}{p}\right)^{\frac{1}{2}}\right]V \qquad k_1=\ m\left[1+\left(\frac{1-m}{m}\frac{1}{q}\right)^{\frac{1}{2}}\right]k \tag{A-17}$$

$$V_2=\ (1-a)\left[1+\left(\frac{a}{1-a}\,p\right)^{\frac{1}{2}}\right]V \qquad k_2=\ (1-m)\left[1+\left(\frac{m}{1-m}\,q\right)^{\frac{1}{2}}\right]Z \tag{A-18}$$

The solution is the concrete form of solution (50) in the main text [see footnote 9].

<u>Step 4.</u> Now, let us transform Fisherian slope (A-1) of the straight line FH into the Marshallian space. The result is

$$\frac{k_2-g}{k_1-g}=\left(\frac{K_1}{K_2}\right)\left(\frac{k_2-K_2}{k_1-K_1}\right) \quad \text{or} \quad \frac{K_2-g}{K_1-g}=\left(\frac{k_1}{k_2}\right)\left(\frac{k_2-K_2}{k_1-K_1}\right) \tag{A-19}$$

These are different from slope (A-2), which stands for the slope of the straight line $F^{\perp}H^{\perp}$ in the Marshallian space. Since $K_1 > K_2$, the first slope of (A-19) between point $F^{\perp}[q]$ and point g is greater than the first slope of (A-2) between point $F^{\perp}[q]$ and point $H^{\perp}[\hat{q}]$. Since $k_1 < k_2$, the second slope of (A-19) between point $H^{\perp}[\hat{q}]$ and g is smaller than the second slope of (A-2) between $H^{\perp}[\hat{q}]$ and point $F^{\perp}[q]$. This implies that the Fisherian straight line FH becomes *kinky line* $F^{\perp}gH^{\perp}$ in the Marshallian space with aid of the transformation and point g is different from point w. Point g plays a pivot role in transformation.

By the same token, let us further transform the Marshallian slope (A-2) of the straight line $F^{\perp}H^{\perp}$ into the Fisherian space to become

$$\frac{V_2-W}{V_1-W}=\left(\frac{Z_1}{Z_2}\right)\left(\frac{V_2-Z_2}{V_1-Z_1}\right) \quad \text{or} \quad \frac{Z_2-W}{Z_1-W}=\left(\frac{V_1}{V_2}\right)\left(\frac{V_2-Z_2}{V_1-Z_1}\right) \qquad \text{(A-20)}$$

These are different from equation (A-1), which stands for the slope of the straight line FH in the Fisherian space. Since $Z_1 < Z_2$, the first slope of (A-20) between point $F[p]$ and point W is smaller than the first slope of (A-1) between point $F[p]$ and point $H[\hat{p}]$. Since $V_1 > V_2$, the second slope of (A-20) between point $H[\hat{p}]$ and point W is greater than the second slope of (A-1) between point $H[\hat{p}]$ and $F[p]$. This implies that the Marshallian straight line $F^{\perp}H^{\perp}$ becomes *kinky line* FWH in the Fisherian space through transformation and point W is different from point G. Point W is a pivot.

The above analysis presents impossibility of the simultaneous straight lines in both spaces. The straight line FH *in the Fisherian space is transformed to become the kinky line* $F^{\perp}gH^{\perp}$ *in the Marshallian space, while the straight line* $F^{\perp}H^{\perp}$ *in the Marshallian space is transformed to become the kinky line* FWH *in the Fisherian space. However, the Fisherian kinky line* $HGWF$ *which kinked twice at* W *and* G *becomes the exact image of the Marshallian kinky line* $H^{\perp}gwF^{\perp}$ *which kinked twice at* w *and* g.

The kinky line $HGWF$ in the Fisherian space is represented by the following two slopes:

$$\frac{V_2-W}{V_1-W}=\left(\frac{Z_1}{Z_2}\right)\left(\frac{V_2-Z_2}{V_1-Z_1}\right), \qquad \frac{Z_2-G}{Z_1-G}=\frac{V_2-Z_2}{V_1-Z_1} \qquad \text{(A-21)}$$

The first one represents the slope of the line segment FW and the second one the slope of the line segment HG. By the same way, the kinky line $H^{\perp}gwF^{\perp}$ in the Marshallian space is represented by the following two slopes :

$$\frac{k_2-w}{k_1-w}=\frac{k_2-K_2}{k_1-K_1}, \qquad \frac{K_2-g}{K_1-g}=\left(\frac{k_1}{k_2}\right)\left(\frac{k_2-K_2}{k_1-K_1}\right) \qquad \text{(A-22)}$$

The first one represents the slope of the line segment $F^{\perp}w$ and the second one the slope of the line segment $H^{\perp}g$. One may verify that first slope of (A-22) is transformed to become the first slope of (A-21) and the second slope of (A-22) is transformed to become the second slope of (A-21).

Step 5. Equations (A-9), (A-10), (A-17), (A-18), and (A-21) in the Fisherian space form the following simultaneous equations system for the kinky phenomenon :

$$V_1 = a\left[1+\left(\frac{1-a}{a}\frac{1}{p}\right)^{\frac{1}{2}}\right]V \qquad Z_1 = c\left[1+\left(\frac{1-c}{c}\frac{1}{\hat{p}}\right)^{\frac{1}{2}}\right]Z$$

$$V_2 = (1-a)\left[1+\left(\frac{a}{1-a}p\right)^{\frac{1}{2}}\right]V \qquad Z_2 = (1-c)\left[1+\left(\frac{c}{1-c}\hat{p}\right)^{\frac{1}{2}}\right]Z$$

$$\frac{V_2-W}{V_1-W} = \left(\frac{Z_1}{Z_2}\right)\left(\frac{V_2-Z_2}{V_1-Z_1}\right) \qquad \frac{Z_2-G}{Z_1-G} = \frac{V_2-Z_2}{V_1-Z_1}$$

Substituting the first four equations into the kinky slope of line segment FW yields

$$(p\hat{p})^{\frac{1}{2}} = \frac{\theta}{aV-cZ}(p)^{\frac{1}{2}} + \left(\frac{a}{1-a}\frac{c}{1-c}\right)^{\frac{1}{2}}\Delta \tag{A-23}$$

where

$$\theta = \left[(1-a)\left(\frac{a}{1-a}\right)^{\frac{1}{2}} - a\left(\frac{c}{1-c}\right)^{\frac{1}{2}}\left(\frac{p}{\hat{p}}\right)^{\frac{1}{2}}\right]V$$
$$+\left[c\left(\frac{a}{1-a}\right)^{\frac{1}{2}} - (1-c)\left(\frac{c}{1-c}\right)^{\frac{1}{2}}\left(\frac{p}{\hat{p}}\right)^{\frac{1}{2}}\right]Z - \left[\left(\frac{a}{1-a}\right)^{\frac{1}{2}} - \left(\frac{c}{1-c}\right)^{\frac{1}{2}}\left(\frac{p}{\hat{p}}\right)^{\frac{1}{2}}\right]W$$

This establishes the kinky relationship between point $F[p]$ and point $H[\hat{p}]$ in the Fisherian space.

The same method may be applied to the Marshallian space with the aid of the simultaneous kinky line phenomenon (A-9), (A-10), (A-17), (A-18), and (A-22) in the Marshallian space to form the following simultaneous equations system.

$$k_1 = m\left[1+\left(\frac{1-m}{m}\frac{1}{q}\right)^{\frac{1}{2}}\right]k \qquad K_1 = n\left[1+\left(\frac{1-n}{n}\frac{1}{\hat{q}}\right)^{\frac{1}{2}}\right]K$$

$$k_2 = (1-m)\left[1+\left(\frac{m}{1-m}q\right)^{\frac{1}{2}}\right]k \qquad K_2 = (1-n)\left[1+\left(\frac{n}{1-n}\hat{q}\right)^{\frac{1}{2}}\right]K$$

$$\frac{K_2-g}{K_1-g} = \left(\frac{k_1}{k_2}\right)\left(\frac{k_2-K_2}{k_1-K_1}\right) \qquad \frac{k_2-w}{k_1-w} = \frac{k_2-K_2}{k_1-K_1}$$

Substituting the first four equations into the kinky slope of line $gH^{\perp}$ yields

$$(q\hat{q})^{\frac{1}{2}} = \frac{\phi}{mk-nK}(q)^{\frac{1}{2}} + \left(\frac{m}{1-m}\frac{n}{1-n}\right)^{\frac{1}{2}}\Omega \tag{A-24}$$

where

$$\phi = \left[(1-m)\left(\frac{m}{1-m}\right)^{\frac{1}{2}} - m\left(\frac{n}{1-n}\right)^{\frac{1}{2}}\left(\frac{q}{\hat{q}}\right)^{\frac{1}{2}}\right]k$$
$$+\left[n\left(\frac{m}{1-m}\right)^{\frac{1}{2}} - (1-n)\left(\frac{n}{1-n}\right)^{\frac{1}{2}}\left(\frac{q}{\hat{q}}\right)^{\frac{1}{2}}\right]K - \left[\left(\frac{m}{1-m}\right)^{\frac{1}{2}} - \left(\frac{n}{1-n}\right)^{\frac{1}{2}}\left(\frac{q}{\hat{q}}\right)^{\frac{1}{2}}\right]g$$

This establishes the kinky relationship between point $F^{\perp}[q]$ and point $H^{\perp}[\hat{q}]$ in the Marshallian space.

The kinky relationship must be identical in both spaces. Let us multiply equation (A-23) and equation (A-24). The result is

$$(p\hat{p}q\hat{q})^{\frac{1}{2}} = \frac{\theta}{aV-cZ}\frac{\phi}{mk-nK}(\hat{p}\hat{q})^{\frac{1}{2}} + \frac{\theta}{aV-cZ}(\hat{p})^{\frac{1}{2}}\left(\frac{n}{1-n}\frac{m}{1-m}\right)^{\frac{1}{2}}\Omega$$
$$+\frac{\phi}{mk-nK}(\hat{q})^{\frac{1}{2}}\left(\frac{a}{1-a}\frac{c}{1-c}\right)^{\frac{1}{2}}\Delta + \left(\frac{a}{1-a}\frac{c}{1-c}\frac{m}{1-m}\frac{n}{1-n}\right)^{\frac{1}{2}}\Delta\Omega \tag{A-25}$$

On the other hand, equations (13), (23), (31), and (36) in the main text yield

$$p\hat{p}q\hat{q} = \frac{a}{1-a}\frac{c}{1-c}\frac{m}{1-m}\frac{n}{1-n} \tag{A-26}$$

Comparing equation (A-25) with equation (A-26) furnishes us

$$\theta = 0 \ , \quad \phi = 0 \tag{A-27}$$
$$\Delta\Omega = 1 \tag{A-28}$$

Equation (A-28) is the same result as equation (61) in the main text even though we use the alternative approach.

Finally, substituting (A-27) and (A-28) into equations (A-23) and (A-24) provides us

$$p\hat{p} = \frac{a}{1-a}\frac{c}{1-c}\Delta^2 \ , \qquad q\hat{q} = \frac{m}{1-m}\frac{n}{1-n}\Omega^2$$

which are equations (53) and (54) in the main text. Q.E.D.