*Subgame Perfect Correlated Equilibria in Repeated Games**

Pavlo Prokopovych† Lones Smith‡

University of Michigan

Economics Department

Ann Arbor, MI 48109-1220

First version - Aug. 2003, This Version -Jan. 2004

Abstract

This paper investigates discounted infinitely repeated games with observable actions extended with an extensive form correlation device. Such games model situations of repeated interaction of many players who choose their individual actions conditional on both public and private information. A number of characterizations of the set of subgame perfect correlated equilibrium payoffs are obtained with the help of a recursive methodology similar to that developed Abreu, Pearce, and Stacchetti (1986, 1990). Notwithstanding the convexity of the set of stage game correlated equilibrium payoffs, we show that the set of subgame perfect correlated equilibrium payoffs need not be convex and may strictly include the set of subgame perfect public randomization equilibrium payoffs.

---

*Both authors thank Ennio Stacchetti for comments that helped us improve the presentation.

†Email: pprokopo@umich.edu;

‡Email: lones@umich.edu; web: www.umich.edu/∼lones. Lones is grateful to the NSF for funding this project.

1 Introduction

This paper investigates discounted infinitely repeated n-player games with observable actions extended with an extensive form correlation device. Such games capture situations of repeated interaction of many players who choose their individual actions conditional on both public and private information.

In repeated games with public monitoring, past public histories relevant to future play are common knowledge at each stage. As a result, such games have a recursive structure and dynamic programming techniques can be successfully employed, with the players' play at each stage being a Nash equilibrium of the corresponding one-shot game. Abreu, Pearce, and Stacchetti (1986, 1990) provide a general framework for the analysis for this class of games. Games where players observe private messages only are considerably more difficult to handle since players' future play depends on their private histories. The introduction of communication via publicly observable announcements among players makes private observations public, thus producing publicly observable history. Kandori and Matsushima (1998) and Compte (1998) used the dynamic programming approach to prove the Folk Theorem for repeated games with private monitoring and communication.

In the present paper, we consider infinitely repeated games with public monitoring where the players observe correlated private messages at the beginning of each stage. Following Aumann (1974, 1987), we assume that private information is generated by an exogenous correlation device. There are two major ways to add a correlation device to a repeated game: first, to add a correlation device acting only once before the beginning of the game, and second, to add a correlation device acting at the beginning of each stage of the game. Forges (1985, 1986) called such devices correlated devices and extensive form correlated devices, respectively, and studied Nash equilibria of the extended games induced by them. In Forges's setting, there is no need to study extensive form correlation devices because both types produce the same set of equilibrium payoffs for two-person undiscounted repeated games with lack of information on one side when the role of the informed player consists exclusively of transmitting information. The same phenomenon was observed by Lehrer (1992) in the context of an undiscounted repeated game with semi-standard information.

The Nash equilibrium concept is not a satisfactory solution concept for extensive form games with observable actions because it permits players to behave irrationally on unreached subgames. In this paper, we explore subgame perfect equilibria of infinitely repeated games extended with an extensive form correlation device. Since such a device remains active throughout the game, we assume that players may condition their play on the history of action profiles played in previous stages and the latest private message they have received from the device. Given a public history, the probability distribution on the product of the players' message sets, according to which the device randomly selects private messages to players, is common knowledge. This is achieved by defining stage k's public history as the realized choices of actions at all stages before k. It is useful to note that any information about private messages sent to players in previous stages or about players' obedience to the device's past recommendations does not affect the way according to which the device selects its current and future recommendations. Another important assumption is that the players' strategies do not depend on past private messages. This assumption leads to the existence of proper subgames and the opportunity to utilize the techniques developed by Abreu, Pearce, Stacchetti (1986, 1990) for studying infinitely repeated games with imperfect monitoring. Thus, the concept of subgame perfect correlated equilibrium is a combination of the correlated equilibrium concept and that of subgame perfectness and is closely related to the concept of subgame perfect publicly correlated equilibrium introduced by Myerson (1991), who studied infinitely repeated games extended with direct public randomization devices and proved a version of the folk theorem for such games.

The extensive form correlation devices we consider send players messages confidentially and separately and are not necessarily direct devices. Proposition 1 asserts that, in infinitely repeated games, subgame perfect correlated equilibria have a simple intertemporal structure, with the players' play at each stage being a correlated equilibrium of the corresponding one-shot game. An important corollary is that the revelation principle holds for such games — any subgame perfect correlated equilibrium payoff can be achieved as a subgame perfect direct correlated equilibrium payoff. Therefore, we can restrict our attention to studying the recursive structure of infinitely repeated games extended with an extensive form direct correlation device and characterizing the set of subgame perfect direct correlated equilibrium payoffs.

In the spirit of dynamic programming, we decompose an equilibrium into an admissible pair that consists of a probability distribution on the product of the players' action sets and a continuation value function. If the distribution is degenerate, we get an admissible pair in the Abreu, Pearce, and Stacchetti sense. This generalization has allowed us to obtain a number of characterizations of the set of subgame perfect correlated equilibrium payoffs similar to those provided by Abreu, Pearce, and Stacchetti (1986, 1990) for games with imperfect monitoring.

In the last section, we present two prisoner's dilemma games. In Example 1, the set of subgame perfect correlated equilibrium payoffs strictly includes not only the set of subgame perfect equilibrium payoffs but also the set of subgame perfect public randomization equilibrium payoffs. In Example 2, the set of subgame perfect direct correlated equilibrium payoffs is not convex, strictly includes the set of subgame perfect equilibrium payoffs, and is strictly contained in the set of subgame perfect public randomization equilibrium payoffs. The latter is possible since, in the presence of a public randomization device, the history of public messages observed in previous stages is also common knowledge at the beginning of each stage, which is not the case when messages are private. Note that, by the revelation principle, the set of subgame perfect correlated equilibrium payoffs of an infinitely repeated game is not necessarily convex as well.

2 Correlation Devices in Finitely Repeated Games: An Example

In this section, we show that the definition of correlated equilibrium given by Aumann (1974, 1987) for normal form games can be readily extended to repeated games.

Consider the following normal form game $G = (N, (A_i)_{i\in N}, (u_i)_{i\in N})$, where $N = \{1, \dots n\}$ is the finite set of players, $A_i = \{1, \dots I_i\}$ is player i's finite set of actions, u_i is player i's payoff function from $A = \underset{i\in N}{\times} A_i$ to the real line R.

For any finite set X, let $\triangle(X)$ denote the set of all probability distributions on X. A correlation device $D = ((M_i)_{i\in N}, \mu)$ consists of a family of finite message sets M_i, one for each player, and a probability distribution μ on the Cartesian product of these message sets $M = \underset{i\in N}{\times} M_i$. When $m = (m_1, \dots, m_n)$ is randomly drawn according to μ, player i is informed about m_i. The extended game G_D is the game where players are first privately informed by the device of their private messages and next play G. A strategy for player i in the extended game is a map f_i from M_i to $\triangle(A_i)$. Let $f_i(a_i \mid m_i)$ denote the probability that player i chooses action a_i if the device's private message to her is m_i, $u_{iD}(f)$ denote player $i's$ expected payoff to the strategy profile $f = (f_1, \dots f_n)$. By definition, $u_{iD}(f) = \sum_{m\in M} u_i(f(m))\mu(m)$, where $u_i(f(m)) = \sum_{a\in A} \prod_{j=1}^{n} f_j(a_j \mid m_j) u_i(a)$. A correlation device $D = ((M_i)_{i\in N}, \mu)$ and a strategy profile $f = (f_1, \dots f_n)$, $f_i : M_i \to \triangle(A_i)$, induce the following probability distribution $\mu_{D,f} \in \triangle(\underset{i\in N}{\times} A_i)$,

$$\mu_{D,f}(a) = \sum_{m\in M} \prod_{j=1}^{n} f_j(a_j \mid m_j)\mu(m),\ a = (a_1, \dots, a_n) \in A.$$

It is obvious that $u_{iD}(f) = \sum_{a\in A} u_i(a)\mu_{D,f}(a)$. A correlated equilibrium (D, f) is a pair consisting of a correlation device D and a strategy profile f such that the strategy profile is a Nash equilibrium of the extended game induced by the device, that is,

$$u_{iD}(f) \geq u_{iD}(g, f_{-i}) \text{ for all } g : M_i \to \triangle(A_i) \text{ and all } i \in N.$$

A direct (canonical) correlation device is one where the set of messages for each player is her action set ($M_i = A_i$ for all $i \in N$). We will identify a direct correlation device $D = ((A_i)_{i\in N}, \mu)$ with the probability distribution μ according to which the device randomly selects private messages for players. In a game extended with a direct correlation device, a strategy for a player is a map from her set of actions to itself. The obedient strategy for player i is the identity map from A_i to A_i. A direct correlated strategy for the players in G is any probability distribution $\mu \in \triangle(A)$. A direct correlated equilibrium is a direct correlated strategy such that the obedient strategy profile forms a Nash equilibrium in the extended game induced by the direct correlation device.

Formally, a direct correlated strategy μ is a direct correlated equilibrium if it satisfies the following incentive constraints:

$$\sum_{a_{-i}\in A_{-i}} \mu(a)(u_i(a) - u_i(e_i, a_{-i})) \geq 0 \text{ for all } i \in N,\ \text{all } a_i \in A_i,\ \text{and all } e_i \in A_i,$$

where $A_{-i} = \underset{j\in N-i}{\times} A_j$, $N - i = \{j \in N : j \neq i\}$.

According to the revelation principle (Myerson, 1982) for normal form games, every correlated equilibrium payoff of G can be achieved as a direct correlated equilibrium payoff of G. Another equivalent version of the principle is that any distribution $\mu_{D,f} \in \triangle(A)$ induced by a correlated equilibrium (D, f) is a direct correlated equilibrium.

As Dhillon and Mertens (1996) have shown, the revelation principle for normal form games fails if, in the definition of correlated equilibrium, the Nash equilibrium concept is replaced with its minimal refinement, the concept of normal form perfect equilibrium. It has turned out that, unlike the set of perfect correlated equilibria, the set of perfect direct correlated equilibria is not necessarily convex. Hence, the set of perfect direct correlated equilibria may be a proper subset of the set of perfect correlated equilibrium distributions.

The basic solution concept we use in this paper is that of subgame perfect correlated equilibrium. As it will be shown below, in infinitely repeated games extended with an extensive form correlation device, the set of subgame perfect direct correlated equilibrium payoffs is not necessarily convex, however the revelation principle holds for such games.

To provide some intuition with respect to possible ways to defining subgame perfect correlated equilibrium strategies, let us assume that the game G is repeated twice, player i's payoff is given by $u_i(a^0) + \delta u_i(a^1)$, where $\delta \in (0, 1)$, a^k is the action profile played at stage k, and the players' stage 0 choices of actions are observable.

A stage-0 direct correlated strategy is some probability distribution $\mu(\cdot \mid h^0) \in \triangle(A)$, where $h^0 = \{\varnothing\}$ is the null history. The correlation device randomly selects $a = (a_1, \ldots, a_n) \in A$ according to $\mu(\cdot \mid h^0)$ and recommends each player i play a_i confidentially and separately.

A stage-1 direct correlated strategy can be described as a map $\mu(\cdot \mid h^1)$ from the set of all possible stage-1 histories $H^1 = A$ to $\triangle(A)$. We also let $H^0 = \{\varnothing\}$. At $h^1 \in H^1$, the correlation device randomly selects an action profile according to $\mu(\cdot \mid h^1)$ and privately makes the corresponding recommendations to the players. Further on, we will identify an extensive form direct correlation device with an extensive form direct correlated strategy.

Thus, an extensive form direct correlated strategy in this twice-repeated game G^2 is a map μ from $H^0 \cup H^1$ to $\triangle(A)$. Intuitively, a direct correlated strategy $\mu : H^0 \cup H^1 \rightarrow \triangle(A)$ is a subgame perfect direct correlated equilibrium of this twice-repeated game if no player can expect to gain by disobeying the device's recommendation after any history of play. In other words, at any stage, the recommendations made by the correlation device should be incentive compatible. In twice-repeated games, we can employ backward induction to ensure incentive compatibility. It should be noted that the problem becomes more complicated in infinitely repeated games. The incentive compatibility of the device's recommendations at each stage will be achieved through extending the techiniques developed by Abreu, Pearce, and Stacchetti (1986, 1990) to infinitely repeated games with an extensive form correlation device.

So, let us write down the system of inequilities ensuring that the device's stage-1 recommendations are incentive compatible for $h^1 \in H^1$:

$$\sum_{a^1_{-i} \in A_{-i}} \mu(a^1 \mid h^1)(u_i(a^1) - u_i(a^1_{-i}, e_i)) \geq 0 \text{ for all } i \in N, \text{ all } a^1_i \in A_i, \text{ all } e_i \in A_i. \tag{1}$$

Assume that μ is common knowledge and these inequalities hold for all $h^1 \in H^1$. Since it is rational for the players to obey the device's recommendations at any $h^1 \in H^1$, player i can compute her stage-1 expected payoff $u^1_i(a^0) = \sum_{a^1 \in A} \mu(a^1 \mid h^1 = \{a^0\}) u_i(a^1)$ for any $a^0 \in A$. The device's recommendations are incentive compatible at stage 0 if $\mu(\cdot \mid h^0)$ is a direct correlated equilibrium of the game $G = (N, (A_i)_{i \in N}, (U_i)_{i \in N})$, where $U_i(a) = u_i(a) + \delta u^1_i(a)$, $a \in A$. Thus, an extensive form direct correlated strategy $\mu : H^0 \cup H^1 \rightarrow \triangle(A)$ is a subgame perfect direct correlated equilibrium of this twice-repeated game if inequalities (1) hold and

$$\sum_{a^0_{-i}\in A_{-i}} \mu(a^0)(U_i(a^0) - U_i(a^0_{-i}, e^0_i)) \geq 0 \text{ for all } i \in N, \text{ all } a^0_i \in A_i, \text{ all } e^0_i \in A_i.$$

If μ is a subgame perfect direct correlated equilibrium, player i's expected

payoff is

$$U_i(\mu) = \sum_{a^0\in A} \mu(a^0 \mid h^0)U_i(a^0) = \sum_{a^1\in A} \mu(a^0 \mid h^0)(u_i(a^0)$$
$$+\delta \sum_{a^1\in A} \mu(a^1 \mid h^1 = \{a^0\})u_i(a^1)) = \sum_{(a^0,a^1)\in A\times A} P(a^0, a^1)(u_i(a^0) + \delta u_i(a^1)),$$

where $P \in \triangle(A \times A)$,

$$P(a^0, a^1) = \mu(a^0 \mid h^0)\mu(a^1 \mid h^1 = \{a^0\}), (a^0, a^1) \in A \times A. \qquad (2)$$

Unfortunately, the dynamic structure of this twice-repeated game would be lost if we tried to define an extensive form direct correlated equilibrium as a Nash equilibrium of the game $\overline{G} = (N, (A_i \times A_i)_{i\in N}, (\overline{U}_i)_{i\in N}), \overline{U}_i(a^0, a^1) = u_i(a^0) + \delta u_i(a^1)$, extended with a direct correlation device randomly selecting its recommendations according to P. The same phenomenon was observed by Myerson (1986) for general multistage games.

Let us assume that $\delta = \frac{1}{2}$ and the stage game is described by the following table:

		Player 2	
	Actions	1	2
Player 1	1	8,8	2,10
	2	10,2	0,0

It is not difficult to check that the extensive form correlated strategy $\mu : H^0 \cup H^1 \to \triangle(A)$, $\mu(\cdot \mid h^0) = (\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}) = (1, 0, 0, 0)$, $\mu(\cdot \mid h^1 = \{(1, 1)\}) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$, $\mu(\cdot \mid (1, 2)) = (0, 0, 1, 0)$, $\mu(\cdot \mid (2, 1)) = (0, 1, 0, 0)$, $\mu(\cdot \mid (2, 2)) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$ is a subgame perfect direct correlated equilibrium of this twice repeated game since the players have no incentives to disobey the mediator's recommendations after any history of play. If each player has followed the recommendation to choose action 1 at stage 0, the stage-1 correlation device selects its recommendations according to $\mu(\cdot \mid (1,1))$, a direct correlated equilibrium of the stage game that gives each player an expected discounted payoff of $3\frac{1}{3}$. If a player deviates at stage 0, she gets two extra utils, but her discounted expected loss is $2\frac{1}{3}$ at stage 1. Another useful observation is that the stage-1 correlated strategies $\mu(\cdot \mid (1,2))$ and $\mu(\cdot \mid (2,1))$ form pure Nash equilibria of the stage game. At the same time, playing the action profile $(1,1)$ at stage 0 is not part of any equilibrium path in pure or mixed strategies.

It is not difficult to check that, in this twice-repeated game, the direct correlated strategy $P \in \triangle(A \times A)$ defined by (2) is not a direct correlated equilibrium of $\overline{G} = (N, (A_i \times A_i)_{i\in N}, (\overline{U}_i)_{i\in N})$, $\overline{U}_i(a^0, a^1) = u_i(a^0) + \delta u_i(a^1)$, $(a^0, a^1) \in A \times A$. Thus, studying subgame perfect correlated equilibria of the twice-repeated game can not be boiled down to studying correlated equilibria of the normal form game $\overline{G}$.

3 Correlation Devices in Infinitely Repeated Games

This section presents the basic properties of infinitely repeated games extended with an extensive form correlation device. By an extensive form correlation device we mean a correlation device that sends separately and confidentially messages to the players at the beginning of each stage. Let $G = (N, (A_i)_{i\in N}, (u_i)_{i\in N})$ be the stage game. Suppose that the game begins at stage 0, with the null history $h^0 = \{\varnothing\}$. At the beginning of stage k, player i observes the public history of past actions actually chosen by the players before stage k $h^k = (a^0, \ldots, a^{k-1})$ and stage k's private recommedation sent her by the correlation device. If $a^k = (a_1^k, \ldots, a_n^k)$ is the action profile chosen at stage $k \in \{0, 1, \ldots\}$, player i's average discounted payoff is:

$$(1-\delta)\sum_{k=0}^{\infty}\delta^k u_i(a^k),$$

where $\delta \in (0,1)$ is the discount factor common to the players. Let $G^{\infty}(\delta)$ denote the discounted infinitely repeated game with observable actions.

An extensive form correlation device $D = ((M_i)_{i\in N}, \mu)$ is an $(n+1)$-tuple consisting of a family of finite message sets M_i and a map μ from $H = \bigcup_{k=0}^{\infty} H^k$ to $\triangle(M)$, where $H^k = (A)^k$ is the set of all possible stage-k public histories, $M = \underset{i\in N}{\times} M_i$. Let $\mu(m = (m_1, \ldots, m_n) \mid h^k)$ denote the conditional probability that the correlation device would send each player i message m_i if the history of past actions is h^k. The extended game $G_D^{\infty}(\delta)$ is the one where, at the beginning of each stage, players are informed by the correlation device D of their private messages and next choose their actions. A strategy for player i is a map f_i from $H \times M_i$ to $\triangle(A_i)$. Here $f_i(a_i \mid h^k, m_i)$ represents the probability that player i plays action a_i at stage k conditional on h^k and m_i. Note player i chooses her action at stage k on the basis of the public information available ($h^k \in H^k$) and the private message she gets from the correlation device at the beginning of the stage. For any correlation device D and strategy profile $f = (f_1, \ldots, f_n)$, we can compute the following: the conditional probability $\mu_{D,f}(a^k \mid h^k)$ that the players would choose action profile a^k if the stage-k public history is h^k

$$\mu_{D,f}(a^k \mid h^k) = \sum_{m\in M}\prod_{i=1}^{n} f_i(a_i^k \mid h^k, m_i)\mu(m \mid h^k);$$

the probability that history $h^{k+1} = \{a^0, \ldots, a^k\}$ has taken place in the first k stages

$$P(h^{k+1} \mid D, f) = \mu_{D,f}(a^0 \mid h^0)\mu_{D,f}(a^1 \mid a^0)\ldots\mu_{D,f}(a^k \mid h^k = \{a^0, \ldots a^{k-1}\});$$

the probability that an action profile a^k would be implemented at stage k

$$P^k(a^k \mid D, f) = \sum_{h^k \in H^k} P(\{h^k, a^k\} \mid D, f);$$

player i's expected average discounted payoff

$$U_i(f \mid \delta, D) = (1-\delta)\sum_{k=0}^{\infty}\delta^k \sum_{a^k \in A} P^k(a^k \mid D, f) u_i(a^k).$$

Having the players' strategy spaces and payoff functions defined, we can extend the definition of correlated equlibrium given in the context of normal form games to infinitely repeated games. An extensive form correlated equilibrium (D, f) of $G^\infty(\delta)$ is a pair consisting of an extensive form correlation device D and a strategy profile f such that the strategy profile is a Nash equilibrium in the extended game induced by the device, that is, $U_i(f \mid \delta, D) \geq U_i(g, f_{-i} \mid \delta, D)$ for all $g : H \times M_i \to \triangle(A_i)$ and all $i \in N$.

Unfortunately, this definition of extensive form correlated equilibrium does not deal successfully with difficulties that may arise with respect to unreached subgames. For games in extensive form, Selten (1965, 1975) proposed a number of refinements of the Nash equilibrium concept. We will employ the concept of subgame perfect equilibrium.

Each stage of the game begins a proper subgame that can be studied in its own. For any correlation device D, strategy profile f, and history h^k, we denote by $D^{|h^k}$ the correlation device $D^{|h^k} = ((M_i)_{i\in N}, \mu^{|h^k})$ induced by D in the subgame starting at h^k, where $\mu^{|h^k}$ is the map from $H = \bigcup_{j=0}^{\infty} H^j$ to $M = \times_{i\in N} M_i$ such that $\mu^{|h^k}(m \mid h^j) = \mu(m \mid h^{k+j} = \{h^k, h^j\})$ for all $m \in M$ and all $h^j \in H$; by $f^{|h^k} = (f_1^{|h^k}, \ldots, f_n^{|h^k})$ the strategy profile induced by f in the subgame starting at h^k, where $f_i^{|h^k}(a_i \mid h^j, m_i) = f_i(a_i \mid h^{k+j} = \{h^k, h^j\}, m_i)$ for all $i \in N$, all $m_i \in M_i$, and all $h^j \in H$; by $\mu^{0|h^k} = \mu^{|h^k}(\cdot \mid h^0)$ the probabity distribution on M according to which the extensive form correlation device

$D = ((M_i)_{i \in N}, \mu)$ selects its stage-k recommendations conditional on history h^k being reached. Note that $D = D^{|h^0}$ and $f = f^{|h^0}$.

For any history h^k and strategy profile $f = (f_1, \ldots, f_n)$, we can compute player i's expected continuation payoff $U_i(f^{|h^k} \mid \delta, D^{|h^k})$ from stage k on:

$$U_i(f^{|h^k} \mid \delta, D^{|h^k}) = (1-\delta)\sum_{j=0}^{\infty}\delta^j \sum_{a^j \in A} P^j(a^j \mid D^{|h^k}, f^{|h^k}) u_i(a^j).$$

An extensive form correlated equilibrium (D, f) of $G^{\infty}(\delta)$ is called *subgame perfect* if $(D^{|h^k}, f^{|h^k})$ is an extensive form correlated equilibrium of $G^{\infty}(\delta)$ for any history $h^k \in H$.

It is not difficult to see that

$$U_i(f^{|h^k} \mid \delta, D^{|h^k}) = \sum_{a \in A} \mu_{D^{|h^k}, f^{|h^k}}(a \mid h^0)((1-\delta)u_i(a) + \delta U_i(f^{|\{h^k, a\}} \mid \delta, D^{|\{h^k, a\}})),$$

Owing to the simple intertemporal structure of G_D^{∞}, there exists a more tractable, equivalent definition of a subgame perfect correlated equilibrium in this game.

Proposition 1. *A pair (D, f) consisting of an extensive form correlation device $((M_i)_{i \in N}, \mu)$ and a strategy profile $f = (f_1, \ldots, f_n)$, $f_i : H \times M_i \to \triangle(A_i)$, is a subgame perfect correlated equilibrium of $G^{\infty}(\delta)$ if and only if, for any history $h^k \in H$, the correlation device $D^{0|h^k} = ((M_i)_{i \in N}, \mu^{0|h^k})$, and the strategy profile $f^{0|h^k} = (f_1^{|h^k}(\cdot \mid h^0), \ldots, f_n^{|h^k}(\cdot \mid h^0))$, $f_i^{|h^k}(\cdot \mid h^0) : M_i \to \triangle(A_i)$, is a correlated equilibrium of the game*

$$G^{0|h^k}(\delta) = (N, (A_i)_{i \in N}, ((1-\delta)u_i(\cdot) + \delta U_i(f^{|\{h^k, \cdot\}} \mid \delta, D^{|\{h^k, \cdot\}})_{i \in N}).$$

Proof. The "only if" part obviously follows from the definition of subgame perfect correlated equilibrium.

Let (D, f) be such that $(D^{0|h^k}, f^{0|h^k})$ is a correlated equilibrium of the game $G^{0|h^k}(\delta)$ for any $h^k \in H$. Assume, by contradiction, there exists a history $h^k \in H$ such that $(D^{|h^k}, f^{|h^k})$ is not an extensive form correlated equilibrium of $G^\infty(\delta)$. Then there exist player i, strategy $g_i : H \times M_i \to \triangle(A_i)$ such that $U_i(g_i, f_{-i}^{|h^k} \mid \delta, D^{|h^k})) > U_i(f^{|h^k} \mid \delta, D^{|h^k})$. For any $K \in \{0, 1, \ldots\}$, define the following function $g_i^K : H \times M_i \to \triangle(A_i)$,

$$g_i^K(h, m_i) = \begin{cases} g_i(h, m_i) \text{ for all } h \in \bigcup_{j=0}^{K} H^j \text{ and all } m_i \in M_i, \\ f_i^{|h^k}(h, m_i) \text{ for all } h \in \bigcup_{j=K+1}^{\infty} H^j \text{ and all } m_i \in M_i. \end{cases}$$

Since the players' future payoffs are discounted, there exists $K \in \{0, 1, \ldots\}$ such that $U_i(g_i^K, f_{-i}^{|h^k} \mid \delta, D^{|h^k}) > U_i(f^{|h^k} \mid \delta, D^{|h^k})$. Player i can not gain by deviating from f_i at stage 0 of the game $G^{|h^{k+K}}(\delta)$ and conforming to f_i thereafter because $(D^{0|h^{k+K}}, f^{0|h^{k+K}})$ is a correlated equilibrium of $G^{0|h^{k+K}}(\delta)$ for any $h^{k+K} \in H$. Therefore, $U_i(g_i^{K-1}, f_{-i}^{|h^k} \mid \delta, D^{|h^k}) > U_i(f^{|h^k} \mid \delta, D^{|h^k})$. Repeating this backward induction argument leads to the conclusion that $U_i(g_i^0, f_{-i}^{|h^k} \mid \delta, D^{|h^k}) > U_i(f^{|h^k} \mid \delta, D^{|h^k})$, which contradicts to the assumption that $(D^{0|h^k}, f^{0|h^k})$ is a correlated equilibrium of the game $G^{0|h^k}(\delta)$. ■

In multi-stage and repeated games, the one-shot deviation principle is an indispensable tool for verifying that a strategy profile is a subgame perfect equilibrium. According to this principle, a strategy profile is a subgame perfect if one-shot deviations are not profitable (any player can not gain by deviating from her strategy in a single stage). It is obvious that Proposition 1 is a version of the one-shot deviation principle.

Corollary 1 *(the one-shot deviation principle for infinitely repeated games extended with an extensive form correlation device). A pair (D, f) consisting of an extensive form correlation device $((M_i)_{i \in N}, \mu)$ and a strategy profile $f = (f_1, \ldots, f_n)$, $f_i : H \times M_i \to \triangle(A_i)$, is a subgame perfect correlated equilibrium of $G^\infty(\delta)$ if and only if the one-shot deviation condition holds: no player can gain by deviating from f in a single stage and conforming to f thereafter.*

Taking into account the importance of the concept of subgame perfect *direct* correlated equilibrium, let us describe some of its most essential elements. An extensive form direct correlation device is an extensive form correlation device $D = ((M_i)_{i \in N}, \mu)$ where $M_i = A_i$ for all $i \in N$. The obedient strategy for player i is a map f_i from $H \times A_i$ to A_i such that $f(h^k, a_i) = a_i$ for all $h^k \in H$ and all $a_i \in A_i$. An extensive form direct correlated equilibrium is just a map μ from H to $\triangle(A)$ such that the obedient strategy profile is a Nash equilibrium of the induced extended game. An extensive form direct correlated equilibrium μ of $G^\infty(\delta)$ is called *subgame perfect* if $\mu^{|h^k}$ is an extensive form direct correlated equilibrium of $G^\infty(\delta)$ for any $h^k \in H$. Let $U_i(\mu^{|h^k}, \delta)$ denote player i's stage-k expected average discounted payoff to an extensive form direct correlated equilibrium μ at h^k.

Proposition 1 implies that the revelation principle holds for infinitely repeated games with extensive form correlation devices.

Corollary 2 *(the revelation principle for infinitely repeated games extended with an extensive form correlation device). Every subgame perfect correlated equilibrium payoff of $G^\infty(\delta)$ can be achieved as a subgame perfect direct correlated equilibrium payoff of $G^\infty(\delta)$.*

Proof. Let (D, f) be a subgame perfect correlated equilibrium of $G^\infty(\delta)$. For any $h^k \in H$, $(D^{0|h^k}\ f^{0|h^k})$ is a correlated equilibrium of the game $G^{0|h^k}(\delta)$. By the revelation principle for normal form games, the probability distribution $\mu_{D^{0|h^k}, f^{0|h^k}}$ on A induced by the correlated equilibrium $(D^{0|h^k}\ f^{0|h^k})$ is a direct correlated equilibrium of $G^{0|h^k}(\delta)$. Consider $\mu : H \to \triangle(A)$, $\mu(a \mid h^k) = \mu_{D^{0|h^k}, f^{0|h^k}}(a)$ for all $h^k \in H$. It follows from Proposition 1 that μ is a subgame perfect direct correlated equilibrium. By construction, the players' expected payoffs to the subgame perfect correlated equilibrium (D, f) coincide with those given by μ. ■

Note that well-known public randomization devices are not extensive form correlation devices in the sense of the definition we have provided. There are two essential differences. First, a public randomization device sends *publicly* observed messages chosen randomly according to a probability distribution at the beginning of each stage and is usually described by a sequence of independent random variables $\Theta_1, \ldots, \Theta_T, \ldots$, each uniformly distributed on $[0, 1]$. Second, the players use random outcomes of the variables to randomize among continuation equilibria, which is possible because the notion of public history is augmented with previously sent public messages, namely the public history at stage k consists of all the action profiles played and the public messages observed before stage k.

Direct public randomization devices were introduced by Myerson (1991), who proved a version of the folk theorem for subgame perfect publicly correlated equilibria of infinitely repeated games with discounting. It is important to notice that, in our formalization, players receive only private messages from an extensive form correlation device. Thus any information about the messages sent before stage k is not common knowledge at the beginning of stage k and can not be used without having to deal with a number of theoretical complications.

The rest of the paper is devoted to studying the set of subgame perfect correlated equilibrium payoffs of $G^\infty(\delta)$. To start with, we prove that this set is not empty.

Proposition 2. *Every infinitely repeated game $G^\infty(\delta)$ has a subgame perfect correlated equilibrium.*

Proof. As Hart and Schmeidler (1989) have shown, every finite normal form game has a correlated equilibrium. Let μ_* be a direct correlated equilibrium of G and μ_*^∞ denote the direct extensive form correlated strategy of $G^\infty(\delta)$ defined as follows:

$$\mu_*^\infty(a \mid h^k) = \mu_*(a) \text{ for all } a \in A \text{ and all } h^k \in H.$$

By the one-shot deviation principle, it is enough to check that no player has incentive to disobey at any $h^k \in H$ and follow the device's recommendations thereafter if its recommendations to the players at all stages are randomly selected according to μ_*^∞. Since

$$U_i(\mu_*^{\infty|h^k}, \delta) = U_i(\mu_*^{\infty|h^0}, \delta) \text{ for all } h^k \in H,$$

and μ_* is a correlated equilibrium of G, no player can gain by disobeying at any $h^k \in H$. ■

As it is shown below, the set of subgame perfect correlated equilibrium payoffs is not necessarily convex. Jumping to the conclusion that this solution concept is flawed is unreasonable since, in many infinitely repeated games, the set of subgame perfect equilibrium payoffs is nonconvex as well (see, for example, Sorin, 1986). Moreover, it has turned out that infinitely repeated games extended with an extensive form correlation device have a recursive structure and the techniques developed by Abreu, Pearce, and Stacchetti (1986, 1990) can be successfully applied in this case.

4 Characterization of the Set of Equilibrium Payoffs

Let V denote the set of feasible and individually (not necessarily strictly) rational payoffs of G, $V_S(\delta)$ the set of subgame perfect equilibrium payoffs of $G^\infty(\delta)$, $V_C(\delta)$ the set of subgame perfect correlated equilibrium payoffs of $G^\infty(\delta)$, $V_P(\delta)$ the set of subgame perfect equilibrium payoffs of $G^\infty(\delta)$ extended with a public randomization device. Further on, we call $V_P(\delta)$ the set of subgame perfect public randomization equilibrium payoffs.

Every subgame perfect equilibrium strategy profile of $G^\infty(\delta)$ can be represented as a subgame perfect direct correlated equilibrium of $G^\infty(\delta)$. Therefore, the set of subgame perfect equilibrium payoffs is a subset of the set of subgame perfect correlated equilibrium payoffs.

Proposition 3. *In $G^\infty(\delta)$, every subgame perfect equilibrium payoff is a subgame perfect direct correlated equilibrium payoff.*

Proof. Let $b = (b_1, \ldots, b_n)$, $b_i : \bigcup_{k=0}^{\infty} H^k \to \triangle(A_i)$, be a subgame perfect equilibrium strategy profile of $G^\infty(\delta)$. We define an extensive form direct correlated strategy $\mu : H \to \triangle(A)$ as follows:

$$\mu(a^k \mid h^k) = b_1(a_1^k \mid h^k) b_2(a_2^k \mid h^k) \ldots b_n(a_n^k \mid h^k) \text{ for all } a^k \in A \text{ and all } h^k \in H.$$

It is not difficult to see that the one-shot deviation condition holds for this extensive form direct correlated strategy. Therefore, the strategy is a subgame perfect direct correlated equilibrium. ■

In the infinitely repeated prisoner's dilemma games studied in Section 5, $V_S(\delta)$ is a proper subset of $V_C(\delta)$. It is useful to note that the set of correlated equilibria payoffs and the set of Nash equilibrium payoffs are identical for a one-shot prisoner's dilemma game.

We are now ready to present the main properties of $V_C(\delta)$, the set of subgame perfect correlated equilibrium payoffs of $G^\infty(\delta)$. By Proposition 2, $V_C(\delta)$ is also the set of subgame perfect direct correlated equilibrium payoffs. Without loss of generality, we assume that any extensive form correlation device added to $G^\infty(\delta)$ is a direct device. Let $M(\delta)$ denote the set of subgame perfect direct correllated equilibrium strategies of $G^\infty(\delta)$.If $\mu \in M(\delta)$, $U(\mu^{|h^k}, \delta) \in V_C(\delta)$ for all $h^k \in H$, and

$$U_i(\mu^{|h^k}, \delta) = \sum_{a^k \in A} \mu(a^k \mid h^k)((1-\delta)u_i(a^k) + \delta U_i(\mu^{|\{h^k, a^k\}}, \delta)).$$

By definition,

$$V_C(\delta) = \{v \in R^n : v = (U_1(\mu^{|h^0}, \delta), \ldots, U_n(\mu^{|h^0}, \delta)) \text{ for some } \mu \in M(\delta)\}.$$

Therefore, if $v \in V_C(\delta)$, there exist a probability distribution $\mu \in \triangle(A)$, a function c from A to $V_C(\delta)$ such that $v = \sum_{a \in A} \mu(a)((1-\delta)u(a) + \delta c(a))$, where $u(a) \triangleq (u_1(a), \ldots, u_n(a))$. This observation is crucial to obtaining a number of powerful characterizations of the set of subgame perfect correlated equilibrium payoffs.

Definition. Given $\delta \in (0,1)$, for any $W \subset R^n$, a pair (μ, c), where μ is a probability distribution on A, c is a function from A to W, is called *admissible* with respect to W if, for all $a_i \in A_i$, $a_j \in A_i \backslash \{a_i\}$, $i \in N$,

$$\begin{aligned} & \sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i})((1-\delta)u_i(a_i, a_{-i}) + \delta c_i(a_i, a_{-i})) \\ \geq & \sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i})((1-\delta)u_i(a_j, a_{-i}) + \delta c_i(a_j, a_{-i})). \end{aligned}$$

For each set $W \subset R^n$, we define

$$\begin{aligned} B_\delta(W) \;\; = \;\; & \{v \in R^n : \; v = \sum_{a \in A} \mu(a)((1-\delta)u(a) + \delta c(a)) \\ & \text{for some pair } (\mu, c) \text{ admissible w.r.t. } W\}. \end{aligned}$$

Any point from $B_\delta(W)$ is called decomposable with respect to W. It is obvious that $V_C(\delta) \subset B_\delta(V_C(\delta))$. Following Abreu, Pearce, and Stacchetti (1990), $W \subset V$ is self-generating if $W \subset B_\delta(W)$. Self-generation is a sufficient condition for a subset of the set of feasible and individually rational payoffs V to be a subset of the set of subgame perfect correlated equilibrium payoffs $V_C(\delta)$.

Proposition 4 (Self-Generation). *If W is self-generating, then* $B_\delta(W) \subset V_C(\delta)$.

Proof. For each $v \in B_\delta(W)$, there exists a pair (μ^0, c^0) admissible w.r.t. W such that $v = \sum_{a^0 \in A} \mu^0(a^0)((1-\delta)u(a^0) + \delta c^0(a^0))$. In its turn, each $c^0(a^0)$, $a^0 \in A$ can be represented as $\sum_{a^1 \in A} \mu^1(a^1 \mid a^0)((1-\delta)u(a^1) + \delta c^1(a^1 \mid a^0))$, where $(\mu^1(\cdot \mid a^0), c^1(\cdot \mid a^0))$ is a pair admissible w.r.t. W. In this way, it is possible to find a pair $(\mu^k(\cdot \mid h^k), c^k(\cdot \mid h^k))$ admissible w.r.t. W for any possible history $h^k = \{a^0, \dots, a^{k-1}\} \in H$. Note that $\mu^k(\cdot \mid h^k)$ is a correlated equilibrium of the game $(N, (A_i)_{i\in N}, ((1-\delta)u_i(\cdot) + \delta c^k(\cdot \mid h^k)_{i\in N})$. It follows from Proposition 1 that the extensive form correlated strategy $\mu : H \to \triangle(A)$,

$$\mu(a \mid h^k) = \mu^k(a \mid h^k) \text{ for all } a \in A \text{ and all } h^k \in H,$$

is a subgame perfect direct correlated equilibrium of $G^\infty(\delta)$ yielding expected discounted average payoff v.■

We now proceed to some properties of the set $V_C(\delta)$.

Corollary 3. *$V_C(\delta)$ is the maximal element (by inclusion) of the collection Γ of all subsets W of V such that $W \subset B_\delta(W)$.*

Proof. Since $V_C(\delta) \subset B_\delta(V_C(\delta))$, the collection Γ is not empty. It is partially ordered by the relation $\subset$. It is not difficult to see that any chain that consists of elements of Γ has an upper bound. By Zorn's lemma, Γ has a maximal element, call it $V(\delta)$. By Proposition 4, $V(\delta) \subset B_\delta(V(\delta)) \subset V_C(\delta)$. Since $V(\delta)$ is the maximal element of Γ, $V_C(\delta)$ is a subset of $V(\delta)$. Therefore, $V(\delta) = V_C(\delta)$.■

The maximality of $V_C(\delta)$ is closely related to the fact that the set is a fixed point of B_δ.

Corollary 4 (Factorization). $V_C(\delta) = B_\delta(V_C(\delta))$.

Proof. By definition, $V_C(\delta) \subset B_\delta(V_C(\delta))$. On the other hand, $B_\delta(V_C(\delta)) \subset V_C(\delta)$ by Proposition 4, hence $V_C(\delta) = B_\delta(V_C(\delta))$.■

It is often useful to remember that $V_C(\delta)$ is a closed subset of V.

Proposition 5. *$V_C(\delta)$ is compact.*

Proof. Let $clV_C(\delta)$ denote the closure of $V_C(\delta)$ and $v^* \in clV_C(\delta)$. Consider a sequence $\{v_j\}_{j=1}^\infty, v_j \in V_C(\delta)$, converging to v^*. For each v_j there exists a pair (μ_j, c_j) admissible with respect to $V_C(\delta)$ such that $v_j =$

$\sum_{a \in A} \mu_j(a)((1-\delta)u(a)+\delta c_j(a))$. For any $a \in A$, the sequence $\{(\mu_j(a), c_j(a))\}_{j=1}^{\infty}$ is bounded, therefore it contains a converging subsequence. Since the set A is finite, there exists a subsequence $\{(\mu_{n_j}, c_{n_j})\}_{n_j=1}^{\infty}$ of $\{(\mu_j, c_j)\}_{j=1}^{\infty}$ such that $\{(\mu_{n_j}(a), c_{n_j}(a))\}_{n_j=1}^{\infty}$converges to some $(\mu^*(a), c^*(a)) \in R^1 \times R^n$ for any $a \in A$.

Without loss of generality, we denote the subsequence again by $\{(\mu_j, c_j)\}_{j=1}^{\infty}$. It is not difficult to see that $\sum_{a \in A} \mu^*(a) = 1$, and $c^*(a) \in clV_C(\delta)$ for all $a \in A$. By continuity we conclude that the pair (μ^*, c^*) is admissible with respect to $clV_C(\delta)$, and, therefore, $clV_C(\delta) \subset B_\delta(clV_C(\delta))$. From the maximal property of $V_C(\delta)$, $V_C(\delta) = clV_C(\delta)$. ■

5 EXAMPLES

Adding an extensive form correlation device to an infinitely repeated prisoner's dilemma game may lead to more efficient outcomes than those that may be achieved with the help of a public randomization device, and moreover, the set of subgame perfect public randomization equilibrium payoffs may be a proper subset of the set of subgame perfect correlated equilibrium payoffs.

Example 1. The stage game is the following prisoner's dilemma game:

		Player 2	
	Actions	1	2
Player 1	1	$1, 1$	$-b, 2$
	2	$2, -b$	$0, 0$

where $b = \frac{2}{5}$. Following Stahl (1991), one can show that, if $\delta \in [\frac{b}{2}, \frac{1}{2})$, the set of subgame perfect public randomization equilibrium payoffs $V_P(\delta)$ is the triangle T with extreme points $(0, 2-b), (2-b, 0)$, and $(0, 0)$. Note that $V_S(\delta) \subset V_P(\delta) = T$ for any $\delta \in [\frac{b}{2}, \frac{1}{2})$. Let us show that there exists a $\delta' \in [\frac{1}{5}, \frac{1}{2})$ such that T is a proper subset of $V_C(\delta)$ for all $\delta \in [\delta', \frac{1}{2})$.

The set W consisting of three points $(\frac{2\delta-b}{1+\delta}, \frac{2-b\delta}{1+\delta})$, $(\frac{2-b\delta}{1+\delta}, \frac{2\delta-b}{1+\delta})$, and $(0,0)$ is self-generating ($W \subset B_\delta(W)$). Proposition 4 implies that W is a subset of $V_C(\delta)$. The reduced normal form game corresponding to the continuation value function $c = (c_{11}, c_{12}, c_{21}, c_{22})$, $c_{11} = c_{22} = (0,0)$, $c_{12} = (\frac{2-b\delta}{1+\delta}, \frac{2\delta-b}{1+\delta})$, $c_{21} = (\frac{2\delta-b}{1+\delta}, \frac{2-b\delta}{1+\delta})$ is as follows:

		Player 2	
	Actions	1	2
Player 1	1	$1-\delta, 1-\delta$	$\frac{2\delta-b}{1+\delta}, \frac{2-b\delta}{1+\delta}$
	2	$\frac{2-b\delta}{1+\delta}, \frac{2\delta-b}{1+\delta}$	$0,0$

Among the Nash equilibria of this game, there are the action profiles $(1,2)$ and $(2,1)$. Therefore, $D = [(\frac{2\delta-b}{1+\delta}, \frac{2-b\delta}{1+\delta}), (\frac{2-b\delta}{1+\delta}, \frac{2\delta-b}{1+\delta})] \subset V_C(\frac{1}{2})$. It is easy to check that the points $(\frac{(1-\delta)b}{\delta}, \frac{2\delta-b}{\delta})$ and $(\frac{2\delta-b}{\delta}, \frac{(1-\delta)b}{\delta})$ may be used as continuations values since they belong to D for any $\delta \in [\frac{1}{5}, \frac{1}{2})$. Below is the reduced normal form game corresponding to $c = (c_{11}, c_{12}, c_{21}, c_{22})$, $c_{11} = c_{22} = (0,0)$, $c_{12} = (\frac{(1-\delta)b}{\delta}, \frac{2\delta-b}{\delta})$, $c_{21} = (\frac{2\delta-b}{\delta}, \frac{(1-\delta)b}{\delta})$:

		Player 2	
	Moves	1	2
Player 1	1	$1-\delta, 1-\delta$	$0, 2-b$
	2	$2-b, 0$	$0,0$

Since the set of Nash equilibria of this game includes the action profiles $(1,2)$, $(2,1)$, and $(2,2)$, we conclude that $T = V_P(\delta) \subset V_C(\delta)$ for any $\delta \in [\frac{1}{5}, \frac{1}{2})$ by Proposition 4.

We now show that there exists a $\binom{v}{v} \in V \backslash T$ that can be supported as a subgame perfect direct correlated equilibrium payoff. Let us take $c = (c_{11}, c_{12}, c_{21}, c_{22})$, $c_{11} = (v,v)$, $c_{12} = (2-b, 0)$, $c_{21} = (0, 2-b)$, $c_{22} = (0,0)$. For the sake of simplicity, we will choose a probability distribution $\mu = (\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22})$ on the set of action profiles such that $\mu_{12} = \mu_{21}$, $\mu_{22} = 0$. The game is symmetric, so let us study it from player 1's point of view. The fact that a point (v,v) can be supported by a pair (μ, c) means

$$v = (1-\delta)(\mu_{11} + (2-b)\mu_{12}) + \delta(\mu_{11}v + (2-b)\mu_{12}) \tag{3}$$

and the following two incentive constraints hold:

$(1-\delta)(\mu_{11} - \mu_{12}b) + \delta(\mu_{11}v + (2-b)\mu_{12}) \geq 2\mu_{11}(1-\delta),$

$2(1-\delta)\mu_{12} \geq (1-\delta)\mu_{12} + \delta\mu_{12}v.$

It follows from the first incentive constraint and (3) that $v \geq 2(1-\delta)(1-\mu_{12})$. At the same time, (3) can be rewritten as follows:

$$v = v(b, \delta, \mu_{12}) = \frac{(1-\delta)\mu_{11} + (2-b)\mu_{21}}{1-\delta\mu_{11}} = 1 - \frac{b\mu_{12}}{1-\delta+2\delta\mu_{12}}.$$

Hence the first incentive constraint holds if δ and μ_{12} are such that

$$1 - \frac{b\mu_{12}}{1-\delta+2\delta\mu_{12}} \geq 2(1-\delta)(1-\mu_{12}).$$

If

$$1 - \frac{b\mu_{12}}{\frac{1}{2}+\mu_{12}} > 1-\mu_{12},$$

then, by continuity, the first incentive constraint holds if $\mu_{12} \neq 0$ and δ is close enough to $\frac{1}{2}$. In other words, for any $\mu_{12} > 0$, there exists a $\delta(\mu_{12}) < \frac{1}{2}$ such that the first incentive constraint holds for all $\delta \in [\delta(\mu_{12}), \frac{1}{2})$. The second incentive constraint is tantamount to the following inequality $v \leq \frac{1-\delta}{\delta}$. This inequality obviously holds if $\delta < \frac{1}{2}$. It is not difficult to see that $v(b, \delta(\mu_{12}), \mu_{12})$ approaches 1 as μ_{12} goes to 0.

Adding a public randomization device to an infinitely repeated game is accompanied by redefining the notion of history, with the stage-k public history including not only the sequence of action profiles $(a^0, a^1, \ldots a^{k-1})$ chosen in previous stages but also the sequence of public messages sent by the device in previous stages. As a result, it is not difficult to find an example where the set of subgame perfect public randomization equilibrium payoffs strictly contains the set of subgame perfect correlated equilibrium payoffs.

In example 2, the set of subgame perfect equilibrium payoffs of an infinitely repeated prisoner's dilemma game is a proper subset of the set of its subgame perfect correlated equilibrium payoffs. The latter set is not convex and, in its turn, is a proper subset of the set of subgame perfect public randomization equilibrium payoffs.

Example 2. We study an infinitely repeated prisoner's dilemma game with observable actions for a fixed discount factor ($\delta = \frac{1}{2}$). The stage game is as follows:

		Player 2	
	Actions	1	2
Player 1	1	1,1	-1,2
	2	2,-1	0,0

Following Sorin (1986), one can check that the set of subgame perfect equilibrium payoffs of this game consists of the square A with extreme points $(0,1)$, $(1,0)$, $(1,1)$, $(0,0)$ and the two line segments $[(1,0),(\frac{3}{2},0)]$, $[(0,1),(0,\frac{3}{2})]$. Let us show that, along with the square, $V_C(\frac{1}{2})$ includes the triangle with extreme points $(\frac{3}{2},0)$, $(0,0)$, $(0,\frac{3}{2})$.

By Proposition 3, the square A is contained in $V_C(\frac{1}{2})$. For the continuation value function $c = (c_{11}, c_{12}, c_{21}, c_{22})$, $c_{11} = c_{12} = c_{21} = (1,1)$, $c_{22} = (0,0)$, the corresponding normal form game is as follows:

		Player 2	
	Actions	1	2
Player 1	1	$1,1$	$0,\frac{3}{2}$
	2	$\frac{3}{2},0$	$0,0$

The set of Nash equilibria of this game includes the action profiles $(1,2)$, $(2,1)$, and $(2,2)$. Therefore, any point of the convex hull of $(0,\frac{3}{2})$, $(0,0)$, $(\frac{3}{2},0)$ is a correlated equlibrium payoff of the game (see Aumann, 1974). Using

Proposition 4, we conclude that the triangle with extreme points $(\frac{3}{2}, 0), (0, 0)$, $(0, \frac{3}{2})$ is contained in $V_C(\frac{1}{2})$.

In contrast to the fact that the set of correlated equilibrium payoffs (see Forges, 1986) is usually convex, the set of payoffs given by subgame perfect correlated equilibrium strategies is not necessarily convex. Let us show that the set $V_C(\frac{1}{2})$ is not convex.

For any point $v \in V_C(\frac{1}{2})$, there exists a pair (μ, c), $\mu = (\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}) \in \triangle(A)$, $c = (c_{11}, c_{12}, c_{21}, c_{22})$, $c_{lm} \in V_C(\frac{1}{2})$, $l = 1, 2,\ m = 1, 2$, such that

$$v = \mu_{11}[\frac{1}{2}\begin{pmatrix} 1 + c^1_{11} \\ 1 + c^2_{11} \end{pmatrix}] + \mu_{12}[\frac{1}{2}\begin{pmatrix} -1 + c^1_{12} \\ 2 + c^2_{12} \end{pmatrix}] + \mu_{21}[\frac{1}{2}\begin{pmatrix} 2 + c^1_{21} \\ -1 + c^2_{21} \end{pmatrix}] + \mu_{22}[\frac{1}{2}\begin{pmatrix} c^1_{22} \\ c^2_{22} \end{pmatrix}]$$

and the following incentive constraints hold

for player 1:

$\mu_{11}(1 + c^1_{11}) + \mu_{12}(-1 + c^1_{12}) \geq \mu_{11}(2 + c^1_{21}) + \mu_{12}c^1_{22}$,

$\mu_{21}(2 + c^1_{21}) + \mu_{22}c^1_{22} \geq \mu_{21}(1 + c^1_{11}) + \mu_{22}(-1 + c^1_{12})$;

for player 2:

$\mu_{11}(1 + c^2_{11}) + \mu_{21}(-1 + c^2_{21}) \geq \mu_{11}(2 + c^2_{12}) + \mu_{21}c^2_{22}$,

$\mu_{12}(2 + c^2_{12}) + \mu_{22}c^2_{22} \geq \mu_{12}(1 + c^2_{21}) + \mu_{22}(-1 + c^2_{21})$.

Let us show that any point of the relative interior of the interval $[(0, \frac{3}{2}), (1, 1)]$ does not belong to $V_C(\frac{1}{2})$. We need to prove that, if $p = (\frac{1}{2}, 1)$, the set

$$V_C(\frac{1}{2}, p) = \{v \in V_C(\frac{1}{2}) : (v, p) = \delta^*(p \mid V_C(\frac{1}{2}))\}$$

consists of two points, namely $(0, \frac{3}{2})$ and $(1, 1)$, where (v, p) is the inner product of v and p, $\delta^*(\cdot \mid V_C(\frac{1}{2})) : R^2 \to R^1$, $\delta^*(p \mid V_C(\frac{1}{2})) \triangleq \max_{v \in V_C(\frac{1}{2})} (p, v)$, is the support function of $V_C(\frac{1}{2})$ (see Rockafellar, 1970). Because

$$(p, \frac{1}{2}(2, -1) + \frac{1}{2}c_{21}) < \delta^*(p \mid V_C(\frac{1}{2})), (p, \frac{1}{2}c_{22}) < \delta^*(p \mid V_C(\frac{1}{2})),$$

for any continuation values c_{21} and c_{22} chosen from V, we conclude that if $v \in V_C(\frac{1}{2}, p)$, then the corresponding $\mu = (\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}) \in \triangle(A)$ is such that $\mu_{21} = \mu_{22} = 0$. Assume that there exists a $v \in V_C(\frac{1}{2}) \cap ((0, \frac{3}{2}), (1,1))$ that can be decomposed with the help of an addmissible pair (μ, c). Since v is different from $(0, \frac{3}{2})$ and $(1,1)$, both μ_{11} and μ_{12} are positive. Therefore, $(p, \frac{1}{2}(-1,2) + \frac{1}{2}c_{12}) = \delta^*(p \mid V_C(\frac{1}{2}))$. Taking also into account the fact that $c_{12} \in V_C(\frac{1}{2}) \subset V$, we conclude that $c_{12} \in [(0, \frac{3}{2}), (1,1)]$, and $c_{12}^2 \geq 1$. Since $\mu_{21} = 0$, the first incentive constraint for player 2 does not hold. Therefore, the set $V_C(\frac{1}{2}, p)$ consists of two points only.

This nonconvexity result is different from that observed when the game is extended with a public randomization device. In the latter case, the set of equilibrium payoffs is the convex hull of $V_S(\frac{1}{2})$ and, therefore, coincides with the set of feasible and individually rational payoffs V.

If messages are public, players can condition their play both on the history of action profiles chosen in previous stages and on the history of past public messages. For example, following Myerson (1991, p. 332), a direct public randomization device is defined as a function μ from $\bigcup_{k=0}^{\infty}(A^k \times A^k)$ to $\triangle(A)$. For any k and any $(a^0, \ldots, a^{k-1}, c^0, \ldots, c^{k-1}) \in A^{2k-2}$, the number $\mu(c^k \mid a^0, \ldots, a^{k-1}, c^0, \ldots, c^{k-1})$ denotes the conditional probability that c^k would be the action profile publicly recommended to the players by the device, if the history of recommendations in previous stages was $(c^0, \ldots, c^{k-1})$ and if the history of action profiles chosen was $(a^0, \ldots, a^{k-1})$. In this case, the public history at the beginning of stage k is $(a^0, \ldots, a^{k-1}, c^0, \ldots, c^{k-1})$. Let us explain how the payoff vector $\frac{1}{2}(0, \frac{3}{2}) + \frac{1}{2}(1,1) = (\frac{1}{2}, \frac{5}{4})$ can be supported in the presence of a public randomization device and why it can not be achieved with an extensive form correlation device. The payoffs $(1,1)$ and $(0, \frac{3}{2})$ are subgame perfect equilibrium ones. To get the payoff $(1,1)$ (the payoff $(0, \frac{3}{2})$), the players play the action profile $(1,1)$ (the action profile $(1,2)$) at stage 0 and choose their future play so that to achieve the following continuation values $c_{11} = (1,1)$, $c_{12} = c_{21} = c_{22} = (0,0)$ ($c_{11} = c_{21} = c_{22} = (0,0)$, $c_{12} = (1,1)$).

The reasoning provided in the proof of Proposition 3 can be employed to determine the corresponding subgame perfect publicly correlated equilibria $\mu^{(1,1)}$ and $\mu^{(0,\frac{3}{2})}$ with the expected payoffs $(1,1)$ and $(0,\frac{3}{2})$, respectively. It should be noted that both $\mu^{(1,1)}$ and $\mu^{(0,\frac{3}{2})}$ are functions from H to $\triangle(A)$ by construction since any information about past recommendations is not used by these devices. The payoff $(\frac{1}{2},\frac{5}{4})$ is obtained if a public randomization device selects its recommendation at stage 0 according to the distribution $\mu((1,1) \mid \varnothing) = \mu((1,2) \mid \varnothing) = \frac{1}{2}$ and acts as the device $\mu^{(1,1)}$ if the recommended action profile is $(1,1)$ or as the device $\mu^{(0,\frac{3}{2})}$ if the recommended action profile is $(1,2)$. It is possible since the information about past recommendations is part of the public history at the beginning of each stage, which is not the case when a correlation device sends players only private messages. Note that, at stage 0, the devices $\mu^{(1,1)}$ and $\mu^{(0,\frac{3}{2})}$ recommend the players play the action profiles $(1,1)$ and $(1,2)$, respectively.

To introduce public randomization effects into the model with an extensive form direct correlation device, it is enough to assume that the device also sends a public message informing the players about the recommended action profile at the end of each stage (after the players have chosen their actions) and redefine the stage-k public history as the sequence of action profiles chosen in previous stages and past public messages. Such a device not only can coordinate players' actions by sending them private messages at the beginning of each stage but also can be used for randomizing among continuation values.

6 REFERENCES

Abreu, D., D. Pearce, and E. Stacchetti (1986). "Optimal Cartel Equilibria with Imperfect Monitoring," *J.Econ.Theory* 39, 251-269.

Abreu, D., D. Pearce, and E. Stacchetti (1990). "Toward a Theory of

Discounted Repeated Games with Imperfect Monitoring," *Econometrica* 58, 1041-1063.

Aumann, R. (1974). "Subjectivity and Correlation in Randomized Strategies," *J.Math.Econ.* 1, 67-96.

Aumann, R. (1987). "Correlated Equilibrium as an Extension of Bayesian Rationality," *Econometrica* 55, 1-18.

Compte, O. (1998). "Communication in Repeated Games with Imperfect Monitoring," *Econometrica* 66, 597-626.

Dhillon A. and J.F. Mertens (1996). "Perfect Correlated Equilibria," *J.Econ.Theory* 68, 279-302.

Forges, F. (1985). "Correlated Equilibria in a Class of Repeated Games with Incomplete Information," *Int.J.Game Theory* 14,129-150.

Forges, F. (1986). "An Approach to Communication Equilibria," *Econometrica* 54, 1375-1385.

Hart, S. and D. Schmeidler D. (1989). "Existence of Correlated Equilibria," *Math.Oper.Res.* 14, 18-25.

Kandori, M. and H. Matsushima (1998). "Private Observation, Communication and Collusion," *Econometrica* 66, 627-652.

Lehrer, E. (1992). "Correlated Equilibria in Two-Player Repeated Games with Nonobservable Actions," *Math.Oper.Res.* 17,175-199.

Myerson, R.B. (1982). "Optimal Coordination Mechanisms in Generalized Principal-Agent Problems," *J.Math.Econ.* 10, 67-81.

Myerson, R.B. (1986). "Multistage Games with Communication," *Econometrica* 54, 323-358.

Myerson, R.B. (1991). Game Theory: Analysis of Conflict, Harvard University Press.

Rockafellar, R.T. (1970) Convex Analysis, Princeton University Press.

Selten, R. (1965). "Spieltheoretische Behandlung eines Oligopolmodells mit Nachfragetragheit," *Zeitschrift fur die gesamte Staatswissenschaft* 12, 301-324.

Selten, R. (1975). "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games," *Int.J.Game Theory* 4,25-55.

Sorin, S. (1986). "On Repeated Games with Complete Information," *Math.Oper.Res.* 11, 147-160.

Stacchetti, E. (2001). "Infinitely Repeated Games," private notes.

Stahl D. II (1991). "The Graph of Prisoners' Dilemma Supergame Payoffs as a Function of the Discount Factor," *Games Econ. Behav.* 3, 368-384.