Education Choice Under Uncertainty

Vincent Hogan*
Department of Economics
University College Dublin

Ian Walker
Department of Economics
University of Warwick

This version: Nov. 21, 2001

Abstract

We apply the theory of real options to the problem of education choice when returns to education are uncertain. We show that the length of time spent in school will be an increasing function of the risk associated with education and not just the expected return. This fact has been neglected in much of the empirical literature on education.

JEL Classification: J24, C61, D81.

Keywords: Education Choice; Dynamic Optimization, Optimal Stopping, Uncertainty.

*Corresponding author: vincent.hogan@ucd.ie; Tel.:+353-1-716-8300; Fax.: +353-1-283-0068; Mail: Dept. of Economics, UCD, Belfield, Dublin 4, Ireland.

1 Introduction

The study of how individuals choose their education levels and the measurement of the economic returns to education has been of great interest since Mincer (1974). This literature has consistently viewed education choice as an investment in human capital; to be thought of in much the same way as we think of investment in physical or financial capital (see Card, 1999, for a comprehensive survey). It is perhaps surprising therefore, that the literature has been relatively silent on the role played by risk in education decisions. A possible reason for this reticence is that we have yet to achieve consensus regarding the measurement of expected returns to education (witness the debate between Card (2001) and Carneiro et. al. (2001) over the appropriate interpretation of IV estimates of the return to schooling). The concept of risk, as measured by the variance of returns, is routinely included in theoretical and empirical discussions of physical and financial investment[1], but is largely absent from discussions of individual schooling choice. In this paper we try to shed some light on the effect of risk on the choice of schooling, bringing the analogy between human capital investment and other investment closer. Indeed we make use of those analytical techniques found to be helpful in the description of financial investment under uncertainty.

While the literature on risk in education is not large, some researchers have analysed the topic. Williams (1979) adapted the optimal portfolio choice model of Merton (1971) to allow for investment in human capital. He defined human capital to be the market value of individual's current stock of skills. The value of the human capital evolved stochastically over time and the individual chose what proportion of her time to allocate to human capital accumulation (as opposed to time allocated to work or leisure). The individual's choice was therefore to allocate time in much the same way as she chooses the weights in a portfolio of financial investments. Thus the problem is very similar to that faced by Merton (1971) and his techniques were applied to its solution.

Williams' (1979) approach has the advantage of, not only allowing us to examine the effects of risk, but also the effect of any covariance of the returns to human capital with the returns to financial assets. However, this approach treats education as occurring continuously and at the same time as work. While this may be true for some sort of on-the-job training, most formal education (and almost all the education studied in typical "returns to education" papers) takes place before formal work and in one go. The problem is not so much one of portfolio choice, but more like an optimal stopping or a tree cutting problem. A reasonable approximation to reality is that most individuals stay in education full time until they judge it optimal

[1] For examples from financial investment see Merton (1971) and Campbell et.al (1997). For physical capital see Caballero and Engle (1999) and Dixit and Pindyck (1994).

to leave, and after leaving, they do not return. This view sees education as a form of irreversible investment – a special case of the classic tree cutting problem. Stochastic versions of this and other optimal stopping problems are analysed in Malliaris and Brock (1982) and Kamien and Schwartz (1991). The application of these techniques to irreversible physical investment (so called “real options”) is surveyed in Dixit and Pindyck (1994). In this paper we apply these techniques to the study of the optimal time to leave school. In essence we view the individual in school as possessing an option to leave at any time and take up work at a wage related (stochastically) to the time spent in school. Once that option is exercised (i.e. the tree is cut down), the individual cannot return to school.

Groot and Oosterbeek (1992) also look at the impact of risk on education choice. They define an increase in risk to be a mean preserving increase in the spread in the distribution of returns, for any arbitrary distribution. This is a very unrestrictive definition of risk, but this generality is achieved at the cost of assuming that the individual is risk neutral. By using the techniques of real options, we can easily accommodate risk aversion but we do so at the cost of limiting ourselves to certain stochastic processes for returns to education. In what follows, we treat the returns to education as being drawn from a normal distribution.

The paper proceeds a follows. Section 2 presents an overview of the problem and the option value approach in the certainty case. Section 3 clarifies exactly how we model stochastic returns. Section 4 solves the model for the simplest case involving uncertain returns. Sections 5 and 6 extend the model to more realistic cases. Section 7 concludes.

2 The Option Approach

Card (2001) provides a model of education choice, where an individual chooses the number of years schooling (S) in order to maximise his or her expected discounted life time utility (1) subject to a budget constraint (2).

$$V = E\left\{ \int_0^S e^{-\rho t} \left\{ u(c_t) + \phi_t \right\} dt + \int_S^T e^{-\rho t} u(c_t) dt \right\} \tag{1}$$

$$\int_0^T e^{-rt} c_t dt = A_0 + \int_S^T e^{-rt} Y_t(s) dt - \int_0^S e^{-rt} F_t dt \tag{2}$$

Assume that the minimum school leaving age is normalised to $t = 0$, lifetime utility is provided by consumption throughout life (i.e. both during and after school) via u, the instantaneous utility function and also by the direct (dis)utility of education, ϕ, where both u and ϕ are increasing concave functions, ρ is the constant rate of time preference, F are school fees, and E is the expectations operator. Education choice is an optimal stopping problem, because the individual faces a once and for all decision to leave school (i.e. choose S) and he or she cannot return at a later date.

Following Card (2001), we will assume that the income process (2) is time separable so that the return measured in terms of lifetime income is the same as the return measured in terms of income over any shorter interval. Formally we have

$$Y_t(s) = W(s)h(t-s)$$

We can think of W as the starting wage after leaving education with S years completed and h as being the factor by which the wage grows in each period as experience and seniority are accumulated ($h(0) = 1$).

Note that this specification of earnings includes two probably unrealistic simplifications. Firstly, Heckman et. al. (2001) have cast doubt on the empirical relevance of the time separability assumption, providing evidence that in the US at least, earnings growth after leaving school is a function of the education level.

Secondly, by making S the choice variable we are saying that earnings are a function of time spent in education and not necessarily the accumulation of formal credentials. Of course the two are closely related, but there is empirical evidence of so-called "sheep-skin" effects i.e. non-linearities in education choice and earnings associated with school and college completion dates.[2] We ignore both issues here as their inclusion would complicate the analysis without shedding much light on the role of risk. But relaxing both the time separability and linear returns assumptions are obvious extensions for future work.

2.1 A Simple Example with Certainty

The solution to the dynamic programme depends crucially on the nature of the budget constraint (2) and subsequent sections of the paper we make it more realistic. For the moment, in order to illustrate the approach we solve a simple example under certainty. To be specific, we assume that $u(c) = c$, and that there is no borrowing or lending, so that $c_t = 0 \ \ \forall \ t < S$ and $c_t = Y_t(s) \ \ \forall \ t \geq S$. We also assume for the moment that there are constant returns (g) to education and that $\rho > g$ (otherwise the agent would never leave school). In order to avoid the other corner solution (leave school immediately) we need to assume that ϕ is constant through time and positive, so that education is valued for its own sake. Finally, we assume that individuals are infinitely lived ($T = \infty$) and that $F_t = 0$, so that neither

[2] See Denny and Harmon (2001).

time nor F are state variables. Thus individuals choose S to maximise

$$V_0 = E\left\{\int_0^S e^{-\rho t}\phi dt + \int_S^\infty e^{-\rho t} Y_t(s) dt\right\} \tag{3}$$

In the absence of uncertainty we have $Y_t(s) = Y_0 \exp(gs)$, where the experience factor $h(t-s)$ has been set equal to one (so that earnings are constant after leaving school) and Y_0 represents earnings with only the minimum schooling. Substituting into the maximand and simplifying we get

$$V_0 = \frac{1}{\rho}\left[Y_0 e^{(g-\rho)S} - \phi e^{-\rho S} + \phi\right]$$

which is an example of the classical tree cutting problem. Differentiating with respect to S we find the optimal level of schooling to be

$$S^* = \frac{1}{g}\ln\left(\frac{\rho}{\rho - g}\frac{\phi}{Y_0}\right)$$

In what follows it turns out to be more convenient to specify the school leaving decision in terms of Y^*, the threshold level of income. The idea is intuitive. While in school the individual keeps an eye on the shadow wage i.e. the wage that he would get were he to leave school immediately. When it reaches a certain critical level, the individual will leave school. By substituting S^* into the expression for $Y_t(s)$ we find the critical value of income to be

$$Y^* = \frac{\rho\phi}{\rho - g} \tag{4}$$

We can then rewrite the expression for S^* to get

$$S^* = \frac{\ln Y^* - \ln Y_0}{g} \tag{5}$$

We can also solve the problem using the real option approach similar to that of Dixit (1993) and Dixit and Pindyck (1994). In the case of certainty, this approach is unnecessarily complicated but it turns out to be the only practical method when returns are stochastic.

The intuition of the option approach is straight forward. At any point in time, while the individual is still in school, he has the option of leaving school. This option itself has value. If he exercises this option he will loose the value of the option (because he cannot return to school in the future) and will receive a life time income that is a function of accumulated schooling. If he chooses not to exercise the option, its value evolves in a manner related to the underlying income process.

More formally, V_t in (3) can be thought of as the value of the option to leave school and start earning income at time t. Assuming that we don't exercise the option (i.e. for $t \in [0..S]$) then we can write equation (6) to describe how V will change over time.[3]

$$pV_t dt = \phi dt + dV_t \tag{6}$$

This Bellman equation (6) can best be understood as an arbitrage equation. The right hand side is the return from staying in school (i.e. holding the option) for length of time dt. It consists of the dividend received over the period (which in our case is the utility derived from education) and the capital gain or loss in the value of the option over the period. Along the optimal path, this return must be equal to the return from the alternative investment strategy of selling the asset and investing the proceeds at the discount rate.

When income reaches a certain level the option is exercised, the individual leaves school and receives that income for life. The present value of this perpetual income stream is Y^*/ρ. Thus at time $t = S$, when the option is about to be exercised, its value will equal Y^*/ρ. If we rewrite V as a function of Y and note that $dV/dt = gY * dV/dY$ then we have (7), where V_Y denotes the derivative and we drop the function's arguments except where necessary.[4]

$$\begin{aligned} pV &= \phi + gYV_Y \\ V(Y^*) &= \frac{Y^*}{\rho} \end{aligned} \tag{7}$$

Equation (7) is a first order ordinary differential equation with one boundary ("value matching") condition. We can verify by substitution that the general solution has the form $V(Y) = BY^\theta + \phi/p$ where $\theta = \rho/g$ and $B = [Y^*]^{1-\theta}/\rho$. The value matching condition explains the constant of integration in terms of the boundary, but we need another condition to tie down the boundary itself (i.e. the system constitutes a "free boundary" problem). This is provided by the "smooth pasting" condition (8).

$$\frac{1}{\rho} = V_Y(Y^*) \tag{8}$$

The smooth pasting condition ensures that the free boundary is chosen optimally. If we stay in school now while the market wage is Y, then we can leave school sometime in the future and earn an even higher wage. The value of this option, when the wage is Y, is given by $V(Y)$. The smooth

[3]We can derive (6) from (3) rigorously using Bellman's Principle of Optimality (see Kamien and Schwartz, 1991, pp. 259-262 , for details).

[4]Throughout the paper we adopt the convention that subscripts indicate (partial) derivatives. Thus V_Y is the first derivative and V_{YY} is second derivative

pasting condition requires that a small change in the threshold income will have no first order effect on the net gain from leaving school. When we leave school we gain (Y/ρ) but loose $V(Y)$. (Note that if the decision to leave school could be reversed at zero cost, there would be no loss). The net gain from leaving school when the (shadow) wage is Y is therefore $Y/\rho - V(Y)$, so the optimal choice of Y^* implies the smooth pasting condition.

Evaluating the derivative in (8) and simplifying using the value matching condition yields the same expression for Y^* as equation (4).

3 Risky Education

Before proceeding to analyse the risky case, we need to clarify what exactly we mean by risky returns to education. In this context we mean that two otherwise identical individuals may end up with different lifetime income profiles, just because of a different draw from the distribution of returns to education. Specifically we model the return to schooling as being drawn from a normal distribution. To keep things simple and to avoid time becoming a state variable, we assume time separability and also that $h(t-s) = 1$.

Consider staying on in school for τ more periods. The return to this extra schooling will equal

$$r(\tau) = \frac{Y(s+\tau) - Y(s)}{Y(s)} \quad \backsim \quad N(g\tau, \tau\sigma^2) \tag{9}$$

which is distributed as a normal random variable with mean g and standard deviation σ when $\tau = 1$. By taking limits of (9) we can show that, in continuous time, the return to a infinitesimally small extra period in school ($r \equiv dY/Y$) will be distributed as $N(gds, \sigma^2 ds)$ implying that Y follows a geometric Brownian motion[5]

$$\frac{r-g}{\sigma} \quad \backsim N(0, ds)$$

or in more usual notation

$$\frac{dY}{Y} = gds + \sigma dz \tag{10}$$

where dz represents the increments of a standard Weiner process i.e. where each increment is drawn from $N(0, ds)$. Equation (10) states that for each instant that the individual remains in school his shadow wage trends up at rate g. In addition at each instant the shadow wage is subject to a (proportionate) shock that has zero mean and variance equal to σ^2. Therefore even if individuals start with the same (deterministic) Y_0 they will end up with

[5] See Dixit (1993) for a derivation of a Brownian motion as the limit of a random walk.

different Y_s. Note that in the absence of uncertainty ($\sigma^2 = 0$) the income process reduces to that analysed in the previous section, i.e. $Y_s = Y_0 \exp(gs)$.

It will sometimes be useful to work in terms of the distribution of Y rather than r. If returns (log wage) are normally distributed then the wage itself will have a log-normal distribution conditional on the initial value.[6]

$$\ln Y_s - \ln Y_0 \backsim N(\mu s, \sigma^2 s)$$
$$\mu = g - \frac{\sigma^2}{2}$$

Geometric Brownian motions have been used elsewhere in economics and operations research to model the prices of other assets.[7] They are analytically convenient and have certain desirable properties. A Brownian motion is a continuous time generalisation of a random walk (see Dixit, 1993) and is therefore fairly intuitive. However, it does have certain specific characteristics that we need to be aware of. Firstly, like a random walk, it is a Markov process. Thus the probability distribution of Y at any time in the future is conditional on its current value and only on its current value. Knowing past values confers no extra information. Secondly the increments of a Brownian motion process that occur over any two non-overlapping intervals are independent. Thirdly these increments are normally distributed with both the mean and variance of a growing linearly with time i.e. $var(r) = \sigma^2 ds \neq (\sigma ds)^2$. Fourthly, the sample path of a Brownian motion is jagged (so no time derivative exists) but continuous (so that no two paths will ever be exactly the same). In fact it can be shown that the Brownian motion is the only process that satisfies these conditions (see Dixit, 1993).

4 Uncertainty: A Simple Case

We assume the same as for the certainty case (i.e. $u(c_t) = Y_t$, $F_t = 0$) except that the shadow wage now follows (10). As before we think in terms of a option problem. Each period the individual has the option to leave school. If she exercises this option she will receive a wage determined by the level of schooling via (10). If she chooses not to exercise the option, she will receive whatever in-school income/utility she has and will wait until next period when she will have the chance to exercise the option again. By this time the wage will have evolved, but the increment will be uncertain when viewed from the previous period. So exercising (or not) the option involves taking a gamble.

The option to leave school is an asset whose value evolves according to a Bellman equation similar to (6) with the added complication that we now

[6]Because of Ito's lemma $dx/x \neq d\ln x$. See Dixit (1993) for a discussion.

[7]See Malliaris and Brock (1982) and Dixit and Pindyck (1994) for surveys and references.

must think in terms of the *expected* capital gain.

$$pV = \phi + \frac{1}{dt}E\{dV\} \tag{11}$$

As before we view V as being a function of Y. Because Y follows a Brownian motion so does V. Using Ito's lemma[8] we can write the stochastic differential for V as

$$dV = \{gYV_Y + \frac{1}{2}\sigma^2 Y^2 V_{YY}\}dt + \sigma Y V_Y dz$$

Note that $E\left[dV\right]$ contains a term in the variance of Y. This has important implications for the effect of risk on decisions. On average shocks have no effect on Y i.e. $E\left[dY\right] = Yg$. However if $V_{YY} > 0$ they will have a positive effect on the change in the value of the option. The reason for this is that if $V_{YY} > 0$ the effect of a negative shock will be smaller in absolute terms than will the effect of positive shocks. The results is that V will trend up (down) because of uncertainty in Y if V_{YY} is positive (negative).

We can substitute dV into the Bellman equation, use the fact that $E[dz] = 0$ and divide by dt to get

$$pV = \phi + gYV_Y + \frac{1}{2}\sigma^2 Y^2 V_{YY} \tag{12}$$

This Bellman equation is very similar to that developed for the case of certain returns with an additional term that reflects the effect of uncertainty in those returns. The equation is a second order non-homogenous ordinary differential equation with a free boundary. We can verify by substitution that the general solution will be

$$V = B_1 Y^{\theta_1} + B_2 Y^{\theta_2} + \phi/\rho \tag{13}$$

where θ_1 is the positive characteristic root and θ_2 is the negative is a root of the fundamental quadratic Q.

$$Q = \frac{1}{2}\sigma^2\theta^2 + (g - \frac{1}{2}\sigma^2)\theta - p \tag{14}$$

Economic theory provides three conditions (15) that determine the two constants of integration and the free boundary.

$$\begin{aligned} \lim_{Y\to 0} V(Y) &= \frac{\phi}{\rho} \\ V(Y^*) &= \frac{Y^*}{\rho} \\ V_Y(Y^*) &= \frac{1}{\rho} \end{aligned} \tag{15}$$

[8] See Malliaris and Brock (1982) for a detailed discussion and Dixit (1993) for a slightly less formal explanation.

The first states that as the shadow wage tends to zero the individual will never leave education and so the value of being in school will simply equal the present value of the direct utility of perpetual education (ϕ/p). This implies that the negative root, θ_2, should have no influence on V as Y tends to zero. If it did then the value of the option to leave school would tend to infinity as income tended to zero. The only way of ensuring this is if $B_2 = 0$.[9]

The second part of (15) is the "value matching" condition. As in the certainty case, it states that the value of the option to leave school is worth, just at the time of leaving school, the present value of lifetime income. This ties down B_1 as a function of the free boundary. The third condition, the "smooth pasting" condition states that for the threshold level of income to be chosen optimally, the net gain to any small changes in Y^* must have only second order effects. Using the value matching and smooth pasting conditions we can solve for Y^*.[10]

$$\begin{aligned} Y^* &= \frac{\theta_1}{\theta_1 - 1}\phi \\ \theta_1 &= \frac{-(g - \frac{1}{2}\sigma^2) + \sqrt{(g - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 p}}{\sigma^2} \end{aligned} \tag{16}$$

As in the case of certainty, sufficient conditions for $Y^* > 0$ are that $\phi > 0$ and $\rho > g$. If the latter were not the case, school would always provide a better return (on average) and it would be optimal to stay in school for ever. Using the implicit function theorem, we can show that Y^* is an increasing function of g and a decreasing function of ρ. Thus high returns to education will cause individuals to stay in school longer whereas a high discount rate will induce them to leave earlier.

We can also show that the threshold value is an increasing function of risk (see appendix) so that the threshold level of income is higher than under certainty. Thus the introduction of risk has caused the individual to delay leaving school. Furthermore using L'Hôpital's rule we can show that (16) reduces to (4) as $\sigma^2 \to 0$. We can also show that Y^* becomes infinite as $\sigma^2 \to \infty$, implying that the agent will never leave school.

Note that risk has an effect on the education decision even though the agent is apparently risk neutral i.e. $u(c) = c$. The reason is that leaving school is an irreversible decision. Therefore there is an incentive to wait for future uncertainty to resolve itself. Or to put it another way, if we stay in school we have the option to leave next period in order to take advantage of

[9]This is a similar to excluding the possibility of a bubble in the price of a financial asset.

[10]For completeness we note that $B_1 = \frac{\phi}{\rho(\theta_1 - 1)}\left(\frac{\theta_1}{\theta_1 - 1}\phi\right)^{-\theta_1}$

a good draw from the distribution of returns or to remain in education so as to avoid a bad draw. So uncertainty increases the potential upside payoff from the option, but, because we will stay in school if the market wage turns out to be low, the downside payoff unchanged. This effect becomes stronger as the riskiness of education increases. Indeed when risk becomes infinite, the agent will never want to exercise the option to leave.

Another way of seeing this is to note that while instantaneous utility is linear ($u(c) = c$), lifetime utility, V, has $V_{YY} > 0$. Again the reason for this is the irreversible nature of the decision to leave school. Heuristically, an individual will tend to respond to positive shock by staying in school so that $V_Y > 0$ when $dY > 0$. But she will tend to react to a negative shock by leaving school so V does not grow i.e. $V_Y = 0$ when $dY < 0$. This asymmetry in the individuals' reaction, is due to the impossibility of returning to school, and generates $V_{YY} > 0$.

Note that the predictions of the model are radically different from those of the portfolio approach of Williams (1979). In his model, an increase in the risk of human capital (or any other asset), would cause the individual to accumulate less of it, other things being equal. In our model, however, an increase in risk will cause an individual to accumulate *more* human capital by staying in school longer. The reason for this difference is the nature of the choice facing the agent. In Williams (1979) she could come in and out of education as she pleases for zero cost. In our model, she cannot return to education once she has decided to leave.

Because the evolution of income is stochastic, there is no expression for S corresponding to (5) for the certainty case. S^* will be a random variable and the best we can do is to describe its distribution. An analytical description of the full distribution is complicated (although see Malliaris and Brock (1982) for an example). Instead we describe it numerically by simulating the system. Let $g = 0.07$, $\sigma = 0.02$ and $\rho = 0.1$ all arbitrary but plausible values. We also set $Y_0 = 10,000$ to represent annual income of those with only compulsory schooling and choose $\phi = 4,565$ to represent net income during school. This value for ϕ is chosen so that the optimal choice of schooling under certainty from (5) is equal to $S^* = 6$. This is arbitrary, but allows us to focus on the effect of introducing uncertainty.

Using these values we can calculate the threshold income (Y^*) under certainty from (4) to be equal to $15,216$. Under uncertainty the corresponding value from (16) of $Y^* = 15,260$ which is less than one percent higher. If the individual starts with income $Y_0 = 10,000$ how long will it take for income to reach the threshold value when it evolves according to (10)? We can work out the distribution function of S^* as in (17) below. The probability that an individual will still be in school at time t (so that S^* greater than t) is equal to the probability that the income process will not have reached the trigger level at time t. Note that $\ln Y_t - \ln Y_0 \sim N(\mu t, \sigma^2 t)$, $\mu = g - \sigma^2/2$

and Φ is the c.d.f. of a standard normal random variable.

$$\begin{aligned} P(S^* \leq t) &= 1 - P(S^* > t) \\ &= 1 - P(\ln Y_t < \ln Y^*) \\ &= 1 - \Phi(Z_t^*) \end{aligned} \tag{17}$$

$$Z_t^* = \frac{\ln Y^* - \ln Y_0 - \mu t}{\sigma\sqrt{t}}$$

Numerical simulation of equation (17) shows that the density of schooling times is approximately normal with mean 6.17 and variance of 0.51. This is not too different from the model with certain returns, reflecting the relatively low level of uncertainty. Figure 1 plots a surface where each cross-section represents the density function and the z axis represents increasing variance of returns. As can be seen, the density of S^* flattens as σ increases, but always remains centered near the certainty value of 6. In order to make this clearer, Table 1 shows the summary statistics for each level of risk.

5 Diminishing Returns

The model of the last section introduced some useful concepts but was unrealistic in several important ways. In particular, constant returns requires the further assumption that school provides positive utility directly in order to avoid a corner solution. In this section will allow for returns to diminish as schooling increases. We now specify income to follow a mean reverting process

$$\frac{dY}{Y} = (\alpha_0 - \alpha_1 Y)dS + \sigma dz$$

This process is similar to the geometric Brownian motion (10) of the last section and we can apply similar techniques. Note that we have specified the return to education to be a diminishing function of the shadow wage and not a of elapsed schooling time. We do this for analytical convenience so as to avoid getting a partial deferential equation with time as a state variable.

As before, the Bellman equation (11) describes the evolution of the value of the option to leave school over the period $[0..S]$. Using Ito's lemma we evaluate the stochastic differential dV. As before notice that the trend rate of growth of V is influenced by variability of Y as well as by the trend in Y.

$$dV = \left\{(\alpha_0 - \alpha_1 Y)V_Y Y + \frac{\sigma^2}{2}Y^2 V_{YY}\right\} dt + \{\sigma V_Y Y\}\, dz$$

Replacing dV in the Bellman equation, dividing across by dt and using $E[dz] = 0$ we get a second order ordinary differential equation similar to

(12) with the exception that we have a slightly more complicated expression in place of g.

$$pV = \phi + (\alpha_0 - \alpha_1 Y)V_Y Y + \frac{\sigma^2}{2} Y^2 V_{YY}$$

It can be verified by substitution that (18) is a general solution to a differential equation of this form where $H(.)$ is the series representation of the confluent hypergeometric function[11] and θ_1 and θ_2 are the positive and negative roots, respectively, of $\frac{\sigma^2}{2}\theta(\theta-1) + \alpha_0\theta - p = 0$.

$$V(Y) = B_1 Y^{\theta_1} H(Y;\theta_1) + B_2 Y^{\theta_2} H(Y;\theta_2) + \phi/p \qquad (18)$$

$$\begin{aligned} H &= 1 + \frac{\theta}{b}x + \frac{\theta(\theta+1)}{b(b+1)}\frac{x^2}{2!} + \frac{\theta(\theta+1)(\theta+2)}{b(b+1)(b+2)}\frac{x^3}{3!}\ldots \\ x &\equiv \frac{2\alpha_1 Y}{\sigma^2} \\ b &\equiv 2\theta + \frac{2\alpha_0}{\sigma^2} \end{aligned}$$

As before we can use the fact that $V(Y) \to \phi/p$ as $Y \to 0$ to set $B_2 = 0$. The value matching and smooth pasting conditions have the same form as (15) and define Y^* and B implicitly. If we solve for Y^* we get (19) which itself must be solved numerically as both H and H_Y are infinite series.

$$(Y^* - \phi)\left[\frac{\theta}{Y^*} + \frac{H_Y}{H}\right] = 1 \qquad (19)$$

Note that the solution to this model incorporates the solution to the simpler model of the last section as a special case. If we eliminate the diminishing returns and set $\alpha_1 = 0$ then $H_Y = 0$ and (19) reduces to (16).

Table 2 shows values of Y^* for certain sample values of α_0, α_1 and σ calculated by numerical simulation of (19). For these simulations we normalize $Y_0 = 1$ and set $\phi = 0$ as with diminishing returns it is no longer needed to avoid a corner solution. We also assume that $\rho = 0.1$. Examination of the table confirms that Y^* is increasing in α_0 and decreasing in α_1. As before, higher returns to education provide an incentive to stay in school. Now we have the additional factor that the return to education is lower at higher levels of education. This provides an incentive to leave education earlier.

We can also see from Table 2 that the threshold level of income is an increasing function of uncertainty. Greater risk will cause the individual to delay leaving school. Again this effect occurs even though the agent is risk neutral, and for the same reason as before– irreversibility in the presence of uncertainty provides an incentive to delay the decision until the uncertainty resolves itself.

[11]See Dixit and Pindyk (1994) page 163 and the referneces cited therein. Note that H reduces to the exponential function when $b = \theta$.

6 Risk Aversion

6.1 Liquidity Constrained

The previous sections assumed that the agent was risk neutral. In this section we allow for individuals to have preferences over risk. For the moment we still impose that there be no borrowing or lending, so that the issue of consumption smoothing does not yet arise. In other words we assume that $c_t = 0 \;\; \forall \; t < S$ and $c_t = Y^* \;\; \forall \; t > S$. In this case it turns out that the solution is more or less the same as in the previous two sections. The Bellman equation is given by (11) as before (we continue to assume that an individual receives net utility of ϕ from participation in education) and so will have the same general solution as before i.e. either (13) or (18) depending on the process for the shadow wage. The form of the utility function only affects utility after leaving school as we have precluded the possibility that the agent may borrow against future income in order to subsidize consumption before graduation.

In fact the only difference between this formulation and the previous section is the boundary conditions. When the individual exercises his option to leave school he will receive lifetime utility equal to Ω. We can calculate this by direct integration assuming that income is constant at Y^*after graduation and assuming that $u(c)$ is CRRA with $u(c) = c^{1-\gamma}/(1-\gamma)$, $\gamma < 1$.

$$\Omega(Y^*) = \int_S^\infty e^{-\rho(t-S)} \frac{c^{1-\gamma}}{1-\gamma} dt = \frac{[Y^*]^{1-\gamma}}{(1-\gamma)\,\rho}$$

As before we assume $V(0) = \phi/\rho$. The value matching and smooth pasting conditions become, $V(Y^*) = \Omega(Y^*)$, and $V_Y(Y^*) = \Omega_Y(Y^*)$ respectively. These have the same interpretation as in the last section, only their form has changed slightly. The resulting threshold income (20) is qualitatively the same as before. All the derivatives of Y^* have the same sign as before; γ just acts as a scaling factor.

$$Y^* = \left[\frac{\phi\theta_1(1-\gamma)}{\theta_1 - (1-\gamma)} \right]^{\frac{1}{1-\gamma}} \tag{20}$$

6.2 Consumption Smoothing

In order to make the problem interesting we need to allow the individual to borrow against future income in order to subsidize consumption while in full time education. The absence of liquidity constraints raises the possibility that an individual will stay in education longer, borrowing to fund consumption during the school years and paying back the debt from higher future earnings. In a world of certainty, the individual's decision making problem can be broken into two steps. First he chooses school in order to maximise lifetime earnings and secondly he allocates these earning through time using the capital markets so as to perfectly smooth consumption. This two stage process breaks down when we combine uncertainty with an irreversibility

The individual maximises lifetime utility V as in (1) repeated below for convenience.

$$V = E\left\{\int_0^S e^{-\rho t}\left\{u(c_t)+\phi_t\right\}dt + \int_S^\infty e^{-\rho t}u(c_t)dt\right\} \tag{21}$$

This maximisation is subject to the budget constraint (2) and the stochastic returns to education (10). We rewrite both in differential form as (22).

$$\begin{aligned} dA_t &= (rA_t - F - c_t)dt \quad \forall \quad t \in [0..S] \\ dA_t &= (rA_t + Y_t - c_t)dt \quad \forall \quad t \in [S..\infty] \\ dY_t &= gY_t dt + \sigma dz \qquad \forall \quad t \in [0..S] \\ dY_t &= \alpha Y_t dt \qquad\qquad \forall \quad t \in [S..\infty] \end{aligned} \tag{22}$$

The first equation in (22) states that individual in school must finance consumption and (constant) school fees, F, by running down asset balances. The second equation states that after graduation asset balances can be rebuilt using earned income. The third equation in (22) shows the evolution of the shadow wage while the individual is in school. As before we assume that the shadow wage evolves according to a geometric Brownian motion so that returns to education are normally distributed and the level of the wage upon graduation is lognormally distributed. The final equation in (22) states that earned income will grow at rate α after leaving school. In order to keep things (relatively) simple we assume that this growth rate is deterministic. Thus $Y_t = Y^* e^{\alpha(t-S)}$ where Y^* is the threshold level of (shadow) wage at when the individual decides to leave school after S periods. Finally, in order to ensure convergence of the integral we assume that p is greater than g, r and α.

Also note that this formulation precludes insurance or any hedging of labour income uncertainty as the only other asset has returns that are not correlated with the returns to education. If we did have perfect insurance against risk in education, the problem would reduce to one of maximising *expected* life-time income and would reduce to that of section 4.

The Bellman equation associated with (21)-(22) is given by (23) where subscripts indicate partial derivatives. Note that the value function, V, is now a function of two state variables, the wage (as before) and also the level of net financial assets.

$$\rho V = \max_c \left\{ u(c) + \phi + V_Y g Y + V_{YY} \frac{\sigma^2}{2} Y^2 + V_A (rA - F - c) \right\} \tag{23}$$

As before we think of school attendance as possession of an option to leave school and earn a salary. The value of this option, V, evolves according to (23). There are some differences with the Bellman equations of previous sections. Firstly, the per period payoff ("dividend") of being in school is now expressed in terms of utility $u(c) + \phi$, where the first term represents the utility of consumption while in school and the second represents the intrinsic utility (or disutility) from being in school. Secondly, school fees (F) must be deducted from the cash available for consumption. The third, and most important difference, is that the individual is able to subsidize consumption while in school by running down asset balances. To this end the individual can choose the level of consumption while in school to maximise lifetime utility or equivalently to maximise the value of the option to quit school.[12] Assuming that the individual will always choose consumption optimally given assets and the wage (i.e. education) then we have the standard first order condition for intertemporal consumption smoothing

$$u_c(c) = V_A$$

If we assume that utility is CRRA, $u(c) = c^{1-\gamma}/(1-\gamma)$, and substitute the first order condition into the Bellman equation, we get equation (24) that describes the stochastic evolution of the option to quit school, conditional on assets and the wage.

$$\rho V = \frac{\gamma V_A^{\frac{\gamma-1}{\gamma}}}{(1-\gamma)} + \phi + gYV_Y + V_{YY} \frac{\sigma^2}{2} Y^2 + V_A (rA - F) \tag{24}$$

We can verify by substitution that the solution to (24) is given by (25) where θ_1 is the positive root of Q in (14) and we have eliminated the negative root in order to impose finite value on the option.

$$\begin{aligned} V(A,Y) &= B_0 (A - F/r)^{1-\gamma} + B_1 Y^{\theta_1} + \phi/\rho \\ B_0 &= \frac{\gamma^\gamma}{(r\gamma - r + \rho)^\gamma (1-\gamma)} \end{aligned} \tag{25}$$

Equation (25) is intuitive. The last two terms are the same as (13), the value function for the simplest case. (Nonetheless the value assigned to the option will be different as B_1 will be different.) The first part of (25) represents

[12] The two are equivalent due to Bellman's Principle of Optimality. See Kamien and Schwartz, 1991, op. cit.

the life-time utility derived from consumption out of financial assets. In effect the introduction of financial assets creates a lower bound for life-time utility. The worst case for the individual is that she never leaves school. In this case she would consume out of assets for ever and enjoy the direct utility of schooling generating a life-time utility of $V = B_0(A - F/r)^{1-\gamma} + \phi/\rho$. Only in the case where the the option to leave school has positive value, will she exercise it at some point, leave school and achieve a life-time utility strictly greater than the lower bound.

When the individual exercises his option and leaves school she will receive a certain salary which will generate a certain lifetime utility, Ω (i.e. the second integral in (21)). The exact value of of post school life-time utility, $\Omega(Y, A)$, depends on how wages evolve after leaving school. Using the usual argument we can construct (26) a Bellman equation for Ω.

$$\rho\Omega = \min_{c} \{u(c) + \Omega_Y \alpha Y + \Omega_A(rA + Y - c)\} \tag{26}$$

This equation is similar to (24) but different in interpretation. The individual once again chooses consumption so as to maximise the value of life time utility condition on assets and the process of income. Here, however, the wage is actually received by the individual as she is working, whereas for equation (24), the Y was the shadow wage i.e. the wage the individual would get the moment he left school. As the individual has left education at this stage, there are no optimal stopping problem and no value matching or smooth pasting conditions. The necessary boundary conditions are provided by the assumption that the integral in (21) converges i.e. life time utility is finite.

If we assume that consumption is optimally chosen after leaving school and that utility is CRRA, then (26) has a solution (27).

$$\Omega(A, Y) = B_0(A + \frac{Y}{r - \alpha})^{1-\gamma} \tag{27}$$

Not surprisingly, given the structure of the problem, (27) has the same form as the first term of (25), but there is a crucial difference between the two. We can view (27) as stating that life-time utility is a function of total wealth, equal to the sum of financial wealth, A, and human capital $Y/(r - \alpha)$. This follows from the assumption that the optimizing individual will borrow against future income in order to smooth consumption after graduation. The situation is different before graduation, however. The human capital term is absent from the first term in (25). The reason is that, strictly speaking, the individual has no marketable human capital, before graduation. What she does have is the option to acquire marketable human capital (by leaving school) at some date in the future. The value of this option appears additively in the value function and not within the parentheses in the same manner as A, because we assume that the option to leave school is an asset which may have value, but nevertheless cannot be traded or used as a collateral for a loan. In that sense there is a liquidity constraint in this problem albeit one that is entirely realistic.

Note also that V is only defined when $rA > F$ i.e. when assets are greater than the present value of future school fees. If this condition is violated then the nature of the problem is fundamentally altered. The reason is that the individual must be able pay her way in school or else she will forced to leave school. The problem is no longer one of optimal stopping as there is no longer a free choice of when to exercise the option. Furthermore, while it may appear from the requirement that $rA > F$ that there is some restriction on borrowing against future income, this is not so. As can be seen from the integral version of the budget constraint (2), the agent is free to borrow and lend unlimited amounts subject only life-time budget balance and the inability to trade the option to leave school.

Finally note that the structure of the problem implies that consumption will jump upon graduation. To see this note that $V_A(A, Y^*) > \Omega_A(A, Y^*)$. The reason is that graduation converts the option (which cannot be traded) into human capital which can, so consumable wealth jumps.

We find the threshold income by imposing the value matching and smooth pasting conditions (28) both of which have the same interpretation as before.

$$\begin{aligned} V(Y^*, A) &= \Omega(Y^*, A) \\ V_Y(Y^*, A) &= \Omega_Y(Y^*, A) \end{aligned} \tag{28}$$

The result is a system of two non-linear simultaneous equations that jointly determine B_1 and Y^* conditional on A and F and the parameters of the model. Eliminating B_1 generates an expression (29) that defines Y^* implicitly.

$$B_0(A - F/r)^{1-\gamma} + \phi/\rho - B_0(A + \tfrac{Y}{r-\alpha})^{1-\gamma} + \tfrac{B_0(1-\gamma)Y}{\theta(r-\alpha)}(A + \tfrac{Y}{r-\alpha})^{-\gamma} = 0 \tag{29}$$

We show in the appendix that we can apply the implicit function theorem to show that Y^* is an increasing function of g and a decreasing function of ρ. Thus high returns to education will cause individuals to stay in school longer whereas a high discount rate will induce them to leave earlier. We can also show that the threshold value is an increasing function of risk.

7 Conclusions

In this paper we apply the techniques of option theory to the study the education decisions of individuals when the returns to education are uncertain. We view an individual in school as possessing an option to leave at any time and take up work at a wage related (stochastically) to the time spent in school. Once that option is exercised, the individual cannot return to school.

We show that high returns to education will cause individuals to stay in school longer whereas a high discount rate will induce them to leave earlier. Furthermore we also show that increasing risk will cause an individual to delay leaving school. This result is not dependent on the risk preferences of agents as it holds for risk neutral agents also. On the face of it, this is curious result, we would expect that higher risk would lead to less investment in human capital. The result stems from treating education as an option. Once the agent leaves school, he can never return. Higher uncertainty, therefore provides an incentive to delay leaving so as to see if uncertainty may resolve itself.

A Solution of Equation (29)

Equation (29) implicitly defines the threshold level of income for the model with consumption smoothing

$$G(Y^*, A, F, \theta) = B_0(A - F/r)^{1-\gamma} + \phi/\rho - B_0(A + \frac{Y^*}{r-\alpha})^{1-\gamma} + \frac{Y^*}{\theta}\frac{B_0}{r-\alpha}(1-\gamma)(A + \frac{Y^*}{r-\alpha})^{-\gamma} = 0$$

We first evaluate the derivatives of G below

$$\begin{aligned} G_{Y^*} &= -B_0\left(A + \frac{Y^*}{r-\alpha}\right)^{-\gamma}(1-\gamma)\frac{(\theta-1)\left(A+\frac{Y^*}{r-\alpha}\right)+\gamma\frac{Y^*}{(r-\alpha)}}{\theta\,(r-\alpha)\left(A+\frac{Y^*}{r-\alpha}\right)} < 0 \\ G_\theta &= -Y^*B_0\,(1-\gamma)\frac{\left(A+\frac{Y^*}{r-\alpha}\right)^{-\gamma}}{\theta^2\,(r-\alpha)} < 0 \\ G_{B_0} &= (A - F/r)^{1-\gamma} - (A + \frac{Y^*}{r-\alpha})^{1-\gamma} + \frac{Y^*}{\theta}\frac{1}{r-\alpha}(1-\gamma)(A + \frac{Y^*}{r-\alpha})^{-\gamma} > 0 \end{aligned}$$

All these derivatives can be signed if we note that $\gamma < 1$, which follows from the definition of CRRA utility, and $\theta_1 > 1$ which follows from the assumption that $\rho > g$.

The above implies that

$$\frac{dY^*}{d\theta} = -\frac{G_\theta}{G_{Y^*}} < 0$$

We also know from applying the implicit function theorem to (14) that θ is an increasing function of ρ and a decreasing function of g and σ.

$$
\begin{aligned}
\frac{d\theta}{d\rho} &= -\frac{Q_\rho}{Q_\theta} = \left[(g - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 p\right]^{-\frac{1}{2}} > 0 \\
\frac{d\theta}{dg} &= -\frac{Q_g}{Q_\theta} = -\theta \left[(g - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 p\right]^{-\frac{1}{2}} < 0 \\
\frac{d\theta}{d\sigma} &= -\frac{Q_\sigma}{Q_\theta} = -\sigma\theta\,(\theta - 1) \left[(g - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 p\right]^{-\frac{1}{2}} < 0
\end{aligned}
$$

So it follows that the threshold income is an increasing function of g and also an increasing function of σ. Finally we note that Y^* is a decreasing function of ρ

$$
\begin{aligned}
\frac{dY^*}{d\rho} &= \frac{dY^*}{d\theta}\frac{d\theta}{d\rho} + \frac{dY^*}{dB_0}\frac{dB_0}{d\rho} + \frac{\partial Y^*}{\partial \rho} \\
&= -\frac{1}{G_{Y^*}}\left[G_\theta \frac{d\theta}{d\rho} + G_{B_0}\frac{dB_0}{d\rho} + G_\rho\right] < 0
\end{aligned}
$$

References

[1] Bertola, G. and Caballero, Ricardo J. (1990), "Kinked Adjustment Costs and Aggregate Dynamics", in Olivier J. Blanchard and Stanley Fischer, eds., *NBER Macroeconomics Annual*, MIT Press, Cambridge, Massachusetts.

[2] Caballero, Ricardo J, and Engel, Eduardo (1999), "Explaining Investment Dynamics in US Manufacturing: A Generalized (S,s) Approach", *Econometrica*, vol. 67, pp. 783-826.

[3] Campbell, John Y. and Lo, Andrew W. and MacKinlay, A. Craig (1997), *The Econometrics of Financial Markets*, Princeton University, Princeton, New Jersey.

[4] Card, David (1999), "The Causal Effect of Education on Earnings", in: Orley Ashenfelter and David Card, eds., *Handbook of Labour Economics*, vol. 3, North-Holland, Amsterdam.

[5] Card, David (2001), "Estimating the Return to Schooling: Progress on Some Persistent Econometric Problems", *Econometrica*, vol. 69, no. 5, pp. 1127-1160.

[6] Card, David and Lemieux, Thomas (2000), "Can Falling Supply Explain the Rising Return to College for Younger Men? A Cohort-Based Analysis." NBER Working Paper No. W7655.

[7] Carneiro, Pedro and Heckman, James J. and Vytlacil, Edward (2001), "Estimating the Return to Education When it Varies Among Individuals", Mimeo, Department of Economics, University of Chicago.

[8] Denny, Kevin and Harmon, Colm (2001), "Testing for Sheepskin Effects in Earnings Equations: Evidence for five Countries", *Applied Economic Letters*, vol. 8, pp. 635-637.

[9] Dixit, Avinash K. (1993), *The Art of Smooth Pasting*, Harwood Academic Publishers, Switzerland.

[10] Dixit, Avinash K. and Pindyck, Robert S. (1994), *Investment Under Uncertainty*, Princeton University, Princeton, New Jersey.

[11] Groot, Wim and Oosterbeek, Hessel (1992), "Optimal Investment in Human Capital under Uncertainty", *Economics of Education Review*, vol. 11, no. 1, pp. 41-49.

[12] Harmon, Colm and Oosterbeek, Hessel and Walker, Ian (2000), "Returns to Education: A Review of Evidence, Issues and Deficiencies in the Literature", discusion paper no. 5, Center for the Economics of Education, London School of Economics and Political Science.

[13] Harmon, Colm and Hogan, Vincent and Walker, Ian (2001), "Dispersion in the Economic Returns to Schooling", Working Paper 16/01, Department of Economics, University College Dublin.

[14] Heckman, James J., Lochner, Lance and Todd, Petra E. (2001), "Fifty Years of Mincer Earnings Regressions." Mimeo, University of Chicago.

[15] Kamien, Morton I. and Schwartz, Nancy L. (1991), *Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management*, Elsevier Science, Amsterdam.

[16] Malliaris, A. G. and Brock, W.A. (1982), *Stochastic Methods in Economics and Finance*, Elsevier Science, Amsterdam.

[17] Merton, Robert C. (1971), "Optimal Consumption and Portfolio Rules in a Continuous-Time Model", *Journal of Economic Theory*, vol. 3, pp. 373-413.

[18] Mincer, Jacob (1974), *Schooling, Experience and Earnings*, Columbia University Press, New York.

[19] Williams, Joseph (1979), "Uncertainty and the Accumulation of Human Capital over the Life Cycle", *Journal of Business*, vol. 52, no. 4, pp. 521-548.

Table 1: Optimal School Leaving

Education Risk σ	Threshold Income Y^*	Time in School (S^*)		
		$E(S^*)$	$Var(S^*)$	Skewness
0	15,216	6.0	0.0	0.0
0.01	15,228	6.09	0.125	0.172
0.02	15,260	6.17	0.511	0.348
0.03	15,314	6.32	1.21	0.518
0.04	15,390	6.52	2.23	0.592
0.05	15,486	6.72	3.35	0.521
0.06	15,604	6.88	4.35	0.409
0.07	15,757	6.98	5.16	0.312
0.08	15,900	7.03	5.87	0.234
0.09	16,077	7.05	6.35	0.185

1. Simulation of basic model as in equation (17)
2. Key parameters: $Y_0 = 10000$; $\rho = 0.1$; $g = 0.07$; $\phi = 4,565$
see text for discussion

Table 2: Threshold Income when there are Diminishing Returns

	α_0	0.15	0.15	0.2	0.2
	α_1	0.025	0.05	0.02	0.01
Risk (σ)					
0.00		2.00	1.00	5.00	10.00
0.02		2.01	1.01	5.01	10.02
0.04		2.02	1.01	5.04	10.08
0.06		2.04	1.02	5.09	10.18
0.08		2.07	1.04	5.16	10.31
0.10		2.1	1.05	5.24	10.47
0.12		2.14	1.07	5.33	10.65
0.14		2.18	1.09	5.43	10.85
0.16		2.23	1.12	5.54	11.08
0.18		2.28	1.14	5.66	11.32
0.20		2.33	1.17	5.79	11.57

1. Simulation of basic model as in equation (19)
2. Key parameters: $Y_0 = 1$; $\rho = 0.1$; $\phi = 0$
see text for discussion

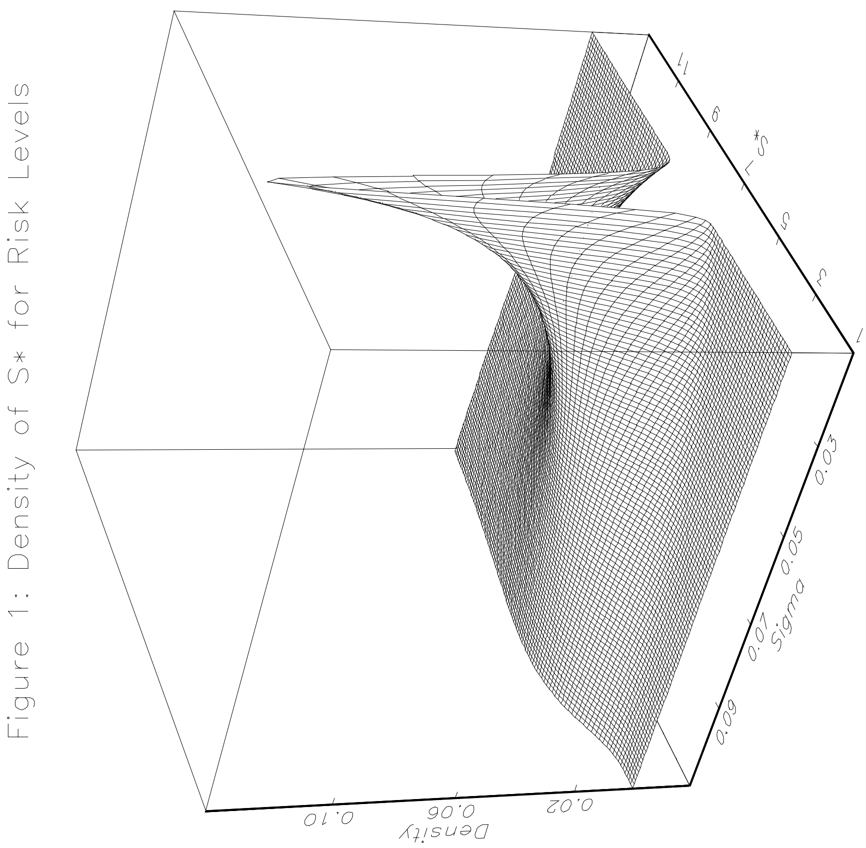
Figure 1: Density of S* for Risk Levels
Density
0.10
0.06
0.02
Sigma
0.09
0.07
0.05
0.03
S*
1
3
5
9
11

Figure 1: