On the Indeterminacy of New-Keynesian Economics

Andreas Beyer[1]
European Central Bank
Postfach 16 03 19
D-60066
Frankfurt am Main
Andreas.Beyer@ecb.int

Roger E. A. Farmer[2]
UCLA, Dept. of Economics
8283 Bunche Hall
Box 951477
Los Angeles, CA 90095-1477
rfarmer@econ.ucla.edu

This version: February 2004

[1]The views expressed in this paper are those of the authors and do not necessarily represent those of the ECB.

[2]The first version of this paper was completed in the summer of 2003 while Farmer was visiting the Directorate General Research as part of the European Central Bank's Research Visitor Programme. He wishes to thank members of DG-Research for their kind hospitality.

Abstract

We study identification in a class of three-equation monetary models. We argue that these models are typically not identified. For any given exactly identified model, we provide an algorithm that generates a class of equivalent models that have the same reduced form. We use our algorithm to provide four examples of the consequences of lack of identification. In our first two examples we show that it is not possible to tell whether the policy rule or the Phillips curve is forward or backward looking. In example 3 we establish an equivalence between a class of models proposed by Benhabib and Farmer [1] and the standard new-Keynesian model. This result is disturbing since equilibria in the Benhabib-Farmer model are typically indeterminate for a class of policy rules that generate determinate outcomes in the new-Keynesian model. In example 4, we show that there is an equivalence between determinate and indeterminate models *even if* one knows the structural equations of the model.

JEL-Classification: C39, C62, D51

Key-words: Identification, indeterminacy.

Non Technical Summary

This paper is about the general lack of identification in linear rational expectations models. It has become common practice in applied monetary economics to estimate single equations using instrumental variables. In a recent paper, Clarida, Gali and Gertler [4] estimate a monetary policy rule and they use their estimated rule to argue that monetary policy before 1980 was very different from policy after 1980. Their work has been criticized in a number of recent papers for failing to pay sufficient attention to the fact that identification is a property of a system. In general one cannot identify a single equation without making assumptions about the nature of other equations in the model. We go beyond this literature by drawing attention to a dimension of the identification problem that is potentially more serious if one hopes to use careful econometrics to help to design economic policies that maximize welfare. We study identification in a class of three-equation monetary models and show that, using data from a single policy regime, it is not possible to tell whether a given period was associated with a policy that was driven purely by fundamental shocks; or whether sunspots also played a role. For any given exactly identified model, we provide an algorithm that generates a class of equivalent models that have the same reduced form. We use our algorithm to provide four examples of the consequences of lack of identification. In our first two examples we show that it is not possible to tell whether the policy rule or the Phillips curve is forward or backward looking. In example 3 we establish an equivalence between a class of models proposed by Benhabib and Farmer [1] and the standard new-Keynesian model. This result is disturbing since equilibria in the Benhabib-Farmer model are typically indeterminate for a class of policy rules that generate determinate outcomes in the new-Keynesian model. In example 4, we show that there is an equivalence between determinate and indeterminate models *even if* one knows the structural equations of the model. In the final section of our paper we draw two conclusions from our analysis. Our first conclusion is that without observing occasional changes in policy it is not possible to tell a good policy rule from a bad one. Our second conclusion is that even if an econometrician observes occasional changes in policy, he still may be unable to tell if the policies that were followed led to determinate or indeterminate equilibria.

1 Introduction

> It is my view, however, that rational expectations is more deeply subversive of identification than has yet been recognized: Christopher A. Sims, "Macroeconomics and Reality", [25], page 7.

This paper is about the general lack of identification in linear rational expectations models. It has become common practice in applied monetary economics to estimate single equations using instrumental variables. In a recent paper, Clarida, Gali and Gertler [4] estimate a monetary policy rule and they use their estimated rule to argue that monetary policy before 1980 was very different from policy after 1980. Their work has been criticized in a number of recent papers for failing to pay sufficient attention to the fact that identification is a property of a system.

In general one cannot identify a single equation without making assumptions about the nature of other equations in the model. Examples of recent papers that make this, or related points, are those of Lindé [16], Lubik and Schorfheide [18], Nason and Smith [21], Mavroedis [19] and Fuhrer and Rudebusch [10]. Our paper has considerable overlap with the work of all of these authors. We go beyond this literature by drawing attention to a dimension of the identification problem that is potentially more serious if one hopes to use careful econometrics to help to design economic policies that maximize welfare. We show that, using data from a single policy regime, it is not possible to tell whether a given period was associated with a policy that was driven purely by fundamental shocks; or whether sunspots also played a role.

We construct an algorithm, implemented in Matlab, that generates equivalence classes of exactly identified models. Using this algorithm we explore a series of examples to illustrate the consequences of the lack of identification in new-Keynesian models of the monetary transmission mechanism. Our paper uses this Matlab algorithm to make four related points.

Our first point is that a central bank might conceptually respond to current past or expected future values of economic variables such as inflation and the output-gap or unemployment. We show that if central bank policy is modeled with an interest-rate reaction function that allows for all of these possibilities then the parameters of this reaction function are not identified. The econometrician cannot tell from the data whether the central bank responds to current, past or future inflation or whether unemployment, for example, enters the policy rule separately from inflation.

Our second point is that the lack of identification extends to the structural equations of the model. The standard new-Keynesian Phillips curve is purely forward looking but Jeffrey Fuhrer [7] has argued that forward looking behavior is unimportant in empirical models of the price process and Jeffrey Fuhrer and Glen Rudebusch [10] claim that forward looking models of the new-Keynesian IS curve also find little support in the data. Following up on this idea, Jordi Gali and Mark Gertler [11] incorporate a Phillips curve that has both forward and backward looking elements into a fully specified structural model of the economy. We demonstrate in our second example that the ability to distinguish forward from backward looking equations is a property of the specification of the entire model. For example, if the central bank responds to lagged inflation, the parameters of the Phillips curve are not identified and the econometrician cannot distinguish a forward looking Phillips curve of the new-Keynesian variety from a backward looking Phillips curve of the kind favored by Robert Gordon [12].

Our third example concerns an equivalence between determinate and indeterminate equilibria.[1] Michael Woodford has argued [28] that a policy

[1]In a previous paper [3], we showed the equivalence between determinate and indeterminate equilibria in a class of one equation models. The current paper fleshes out this

maker should strive to design a rule that excludes multiple indeterminate equilibria. In simple new-Keynesian models this often implies that monetary policy should be *active* in the sense that a policy maker should raise the nominal interest rate by more than 1% in response to a 1% increase in expected inflation. Benhabib and Farmer [1] have proposed an alternative theory of the monetary transmission mechanism in which real balances enter the production function. In their model the standard wisdom about active policy is reversed. If monetary policy is active, equilibrium is *indeterminate*. In our third example we demonstrate that a new-Keynesian model with a determinate equilibrium has the same reduced form as a Benhabib-Farmer economy in which equilibrium is indeterminate. This example is disturbing since it suggests that an econometrician may be unable to distinguish a good policy from a bad one.

In a recent working paper [2] we proposed a method of identifying the structural equations of a rational expectations model: Our method relies on a natural experiment. Using this approach, a central bank might be able to identify a set of equations that would enable it to correctly predict the private sector response to changes in its policy rule. This idea leads us to our fourth and final experiment.

In experiment 4, we distinguish the information set of the policy maker from that of the econometrician. We assume that the econometrician and the policy maker both know the values of the deep parameters of the private sector. To acquire this knowledge, perhaps the policy maker conducted an experiment in which he changed the policy rule at a discrete point in time. An experiment of this kind would allow both the policy maker and the econo-

example by providing a multi-equation analog in the context of alternative theories of the monetary transmission mechanism. The idea that determinate and indeterminate models may be observationally equivalent has been discussed in the case of calibrated examples by Takashi Kamihigashi [13] and Harold Cole and Lee Ohanian [5].

metrician to obtain consistent estimates of a subset of the deep structural parameters of the economy: this was the main point of our earlier paper [2]. The policy maker and the econometrician are distinguished by the assumption that the policy maker knows the rule that was followed before and after the change in policy but the econometrician does not. We argue that this is a reasonable assumption because the policy maker knows his own actions but the econometrician cannot infer these actions by observing the outcome of the policy-maker's experiment.

We use experiment 4 to show that the econometrician cannot distinguish two different rules, one of which was associated with a determinate equilibrium and the other with an indeterminate equilibrium. Even if the econometrician knows the deep structural parameters he cannot infer from observing the outcome of a regime change whether the policies before and after the switch were determinate or indeterminate.

In the final section of our paper we draw two conclusions from our analysis. Our first conclusion is that without observing occasional changes in policy it is not possible to tell a good policy rule from a bad one. Our second conclusion is that even if an econometrician observes occasional changes in policy, he still may be unable to tell if the policies that were followed led to determinate or indeterminate equilibria.

2 A Class of Linear Models

2.1 The Structural Form

We consider the following class of linear rational expectations models,

$$
\begin{aligned}
AY_t + FE_t\left[Y_{t+1}\right] &= B_1 Y_{t-1} + B_2 E_{t-1}\left[Y_t\right] + C + \Psi_v V_t, && (1)\\
E_t\left[V_t V_s'\right] &= \begin{cases} I, & t = s, \\ 0, & \text{otherwise} \end{cases} && (2)
\end{aligned}
$$

In this notation A, F, Ψ_v B_1 and B_2 are $l \times l$ matrices of coefficients, C is an $l \times 1$ matrix of constants, E_t is a conditional expectations operator and $\{V_t\}$ is a weakly stationary i.i.d. stochastic process with covariance matrix I and mean zero.[2] Lowercase letters are scalars, and uppercase letters represent vectors or matrices. We maintain the convention that coefficients of endogenous variables appear on the left side of each equation with positive signs and explanatory variables appear on the right side of equations with positive signs.

2.2 The Companion Form

We define the *companion form* of Equation (1) as follows

$$
\overset{\tilde{A}_0}{\begin{bmatrix} A & F \\ I & 0 \end{bmatrix}} \overset{X_t}{\begin{bmatrix} Y_t \\ E_t\left[Y_{t+1}\right] \end{bmatrix}} = \overset{\tilde{A}_1}{\begin{bmatrix} B_1 & B_2 \\ 0 & I \end{bmatrix}} \overset{X_{t-1}}{\begin{bmatrix} Y_{t-1} \\ E_{t-1}\left[Y_t\right] \end{bmatrix}} + \overset{\tilde{C}}{\begin{bmatrix} C \\ 0 \end{bmatrix}} \tag{3}
$$
$$
+ \overset{\tilde{\Psi}_v}{\begin{bmatrix} \Psi_v \\ 0 \end{bmatrix}} V_t + \overset{\tilde{\Psi}_w}{\begin{bmatrix} 0 \\ I \end{bmatrix}} W_t,
$$

or, more compactly:

$$
\tilde{A}_0 X_t = \tilde{A}_1 X_{t-1} + \tilde{C} + \tilde{\Psi}_v V_t + \tilde{\Psi}_w W_t. \tag{4}
$$

[2] We will focus on the case of one lag, but our method can easily be expanded to include additional lags or additional leads of expected future variables.

This is a system of $2 \times l$ equations in $2 \times l$ endogenous variables; these are the l variables Y_t and the l date t expectations of Y_{t+1}, denoted $E_t [Y_{t+1}]$. The structural form is a system of l equations in these $2 \times l$ endogenous variables. To close this model we assume that the subjective probability $E_t [Y_{t+1}]$ is taken with respect to the true conditional distribution of the variables Y_{t+1}. This assumption is coded into the second l rows of Equation (3) which defines the $l \times 1$ vector of non-fundamental errors W_t to be equal to the difference between the realization of Y_t and its date $t-1$ expectation.

2.3 The Reduced Form

The reduced form of an econometric model is a set of equations, one for each endogenous variable, that explains each endogenous variable as a function of exogenous and predetermined variables. In models in which expectations do not appear, one would compute the reduced form by premultiplying Equation (3) by $\tilde{A}_0^{-1}$. In rational expectations models this method breaks down for two reasons. First, it may be the case that both $\tilde{A}_0$ and $\tilde{A}_1$ are singular; second, even if $\tilde{A}_0$ is non-singular, the roots of $\left(\tilde{A}_0^{-1}\tilde{A}_1\right)$ (the matrix that premultiplies the lagged variables in the reduced form) will not in general all be inside the unit circle.

The reduced form of Equation (1) is given by the following equation,

$$X_t = \Gamma^* X_{t-1} + C^* + \Psi_V^* V_t + \Psi_W^* W_t, \tag{5}$$

which can be expanded as,

$$\begin{bmatrix} Y_t \\ E_t [Y_{t+1}] \end{bmatrix} = \begin{bmatrix} \Gamma_{11}^* & \Gamma_{12}^* \\ \Gamma_{21}^* & \Gamma_{22}^* \end{bmatrix} \begin{bmatrix} Y_{t-1} \\ E_{t-1} [Y_t] \end{bmatrix} + \begin{bmatrix} C_1^* \\ C_2^* \end{bmatrix} + \begin{bmatrix} \Psi_{1V}^* \\ \Psi_{2V}^* \end{bmatrix} V_t + \begin{bmatrix} \Psi_{1W}^* \\ \Psi_{2W}^* \end{bmatrix} W_t.$$

In practice the reduced form can be computed in a number of ways. Christopher Sims [26] provides Matlab code that uses a QZ decomposition to find the solution for a model written in companion form when the solution is unique.[3] The QZ decomposition uses the generalized eigenvalues of $\left\{\tilde{A}_0, \tilde{A}_1\right\}$ and it does not require either matrix to be non-singular. There are three possible cases to consider when deriving this solution. Case (1) is that there exists a unique equilibrium. It is also possible that (2) there is no stationary equilibrium or (3) there are multiple stationary indeterminate equilibria.

If the model has a unique determinate solution, the non-fundamental errors W_t are endogenously determined as functions of the fundamental errors V_t and they do not enter the reduced form, it follows that the matrix Ψ^*_W is identically zero. If there are insufficient unstable roots to uniquely determine the values of the endogenous variables the solution is said to be indeterminate. In this case, the matrices Ψ^*_{1W} and Ψ^*_{2W} each have column rank equal to the degree of indeterminacy.

In Section 6, we study a series of examples in which Y_t has dimension 3. For these examples, determinacy means that the companion form of the system has three unstable eigenvalues. If there are only two unstable eigenvalues there is one degree of freedom in choosing the vector W_t and one can think of the variables W_t as independent 'sunspot' shocks. Similarly if the number of unstable eigenvalues is equal to 1 or 0 there are respectively 2 or 3 degrees of indeterminacy. In theses cases there is the potential for 2 or 3 independent sunspot shocks to influence the behavior of the endogenous variables.

[3] Thomas Lubik and Frank Schorfheide [17] discuss the case when the solution is indeterminate.

3 Identification

In this section we show how to construct equivalence classes of structural models that have the same reduced form. We begin with a given structural model that we parameterize with a vector θ and we construct its reduced form. Given this reduced form, we add a set of linear restrictions $\{R, r\}$ that exactly identifies some possibly different model that we parameterize with a vector $\bar{\theta}$. These linear restrictions are encoded into a matrix R and a vector r with the property $R\bar{\theta} = r$.

The structural form of the model is given by the expression

$$\begin{bmatrix} A & F \end{bmatrix} \begin{bmatrix} Y_t \\ E_t\left[Y_{t+1}\right] \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} Y_{t-1} \\ E_{t-1}\left[Y_t\right] \end{bmatrix} + C + \Psi_V V_t. \tag{6}$$

From the data one can recover estimates of Γ^* and C^*, where

$$X_t = \Gamma^* X_{t-1} + C^* + \Psi_V^* V_t + \Psi_W^* W_t. \tag{7}$$

In all of the cases we consider in this paper, the true data generation process is determinate: This implies that the sunspot variables W_t do not enter the reduced form. We assume in each example that we study that the covariance matrix of V_t is the identity matrix I_l. This assumption is unrestrictive since any pattern of correlations in the shocks can be encoded into the impact matrix Ψ_V. The parameters of the variance-covariance matrix of V_t and the matrix of impact coefficients Ψ_V are not separately identified.

Premultiplying (7) by $\begin{bmatrix} A & F \end{bmatrix}$ and equating coefficients leads to the following system of $l\,(3l+1)$ equations

$$\underset{l\times 2l}{\begin{bmatrix} \underset{l\times l}{A} & \underset{l\times l}{F} \end{bmatrix}} \underset{2l\times(3l+1)}{\begin{bmatrix} \underset{2l\times 2l}{\Gamma^*} & \underset{2l\times 1}{C^*} & \underset{2l\times l}{\Psi_V^*} \end{bmatrix}} = \underset{l\times(3l+1)}{\begin{bmatrix} \underset{l\times l}{B_1} & \underset{l\times l}{B_2} & \underset{l\times 1}{C} & \underset{l\times l}{\Psi_V} \end{bmatrix}}. \tag{8}$$

Let θ be the parameter vector that contains the structural parameters of the DGP. This consists of the elements of the five $(l \times l)$ matrices A, F, B_1, B_2 and Ψ_V and the $l \times 1$ elements of the vector C. After re-arranging Equation (8) and exploiting the properties of the Kronecker product, this system can be written as

$$\overbrace{\underset{l(3l+1)\times l(5l+1)}{\left(\underset{l\times l}{[I_l]} \otimes \underset{(3l+1)\times(5l+1)}{\begin{bmatrix} \underset{((3l+1)\times 2l)}{\begin{matrix} \underset{2l\times 2l}{\Gamma^{*\prime}} \\ \underset{1\times 2l}{C^{*\prime}} \\ \underset{l\times 2l}{\Psi_V^{*\prime}} \end{matrix}} & \vdots & \underset{(3l+1)\times(3l+1)}{-I_{(3l+1)}} \end{bmatrix}} \right)}}^{H} \overbrace{\underset{l(5l+1)\times 1}{\left(vec \underset{(5l+1)\times l}{\begin{bmatrix} \underset{l\times l}{A'} \\ \underset{l\times l}{F'} \\ \underset{l\times l}{B_1'} \\ \underset{l\times l}{B_2'} \\ \underset{1\times l}{C'} \\ \underset{l\times l}{\Psi_V'} \end{bmatrix}} \right)}}^{\theta} = \overbrace{\underset{l(3l+1)\times 1}{[0]}}^{h},$$

or more compactly,

$$\underset{l(3l+1)\times l(5l+1)}{H(\theta)} \underset{l(5l+1)\times 1}{\theta} = \underset{l(3l+1)\times 1}{h},$$

which is a system of $l(3l+1)$ equations in the $l(5l+1)$ parameter vector θ. This parameter vector uniquely determines the reduced form H.

We refer to the mapping from the structural parameters θ to the reduced form parameters H as $M^{DGP}(\theta)$ and we write

$$H = M^{DGP}(\theta).$$

We fix the reduced form matrix H and we search for some (possibly different) parameter vector $\bar{\theta}$ that satisfies the *linear* equations

$$H\bar{\theta} = h. \tag{9}$$

Equation (9) consists of $l\,(3l+1)$ linear equations in the $l\,(5l+1)\times 1$ parameter vector $\bar{\theta}$. Exact identification requires an additional $2l^2$ independent linear restrictions which we write as

$$\underset{l(2l)\times l(5l+1)}{R}\ \underset{l(5l+1)\times 1}{\bar{\theta}} = \underset{l(2l)\times 1}{r}. \tag{10}$$

Our algorithm takes as inputs, θ, R and r and it generates a new parameter vector $\bar{\theta}$,

$$\bar{\theta} \equiv vec\left[\left(\bar{A}', \bar{F}', \bar{B}_1', \bar{B}_2', \bar{C}', \bar{\Psi}_V', \bar{\Psi}_W'\right)'\right],$$

that has the same reduced form, H, as the DGP: We refer to the mapping from the new parameter vector $\bar{\theta}$ to the reduced form parameters H as $M^{EQV}\left(\bar{\theta}\right)$ and we write

$$H = M^{EQV}\left(\bar{\theta}\right) = M^{DGP}\left(\theta\right).$$

We call $M^{EQV}\left(\bar{\theta}\right)$ the *equivalent* model.

Stacking equations (9) and (10) leads to the system

$$\left(\left[\begin{array}{c} \left[\underset{l\times l}{I_l}\right] \otimes \underset{l(3l+1)\times l(5l+1)}{\left(\underset{(3l+1)\times(5l+1)}{\left[\underset{((3l+1)\times 2l)}{\begin{array}{c}\underset{2l\times 2l}{\Gamma^{*\,\prime}} \\ \underset{1\times 2l}{C^{*\,\prime}} \\ \underset{l\times 2l}{\Psi_V^{*\,\prime}}\end{array}} \ \vdots - I_{(3l+1)}\right]}\right)} \\ \underset{l(2l)\times l(5l+1)}{R} \end{array}\right]_{l(5l+1)\times l(5l+1)}\right) \left(vec \underset{(5l+1)\times l}{\left[\begin{array}{c} \underset{l\times l}{\bar{A}'} \\ \underset{l\times l}{\bar{F}'} \\ \underset{l\times l}{\bar{B}_1'} \\ \underset{l\times l}{\bar{B}_2'} \\ \underset{1\times l}{\bar{C}'} \\ \underset{l\times l}{\bar{\Psi}_V'} \end{array}\right]}\right) = \underset{l(5l+1)\times 1}{\left(\begin{array}{c} \underset{l(3l+1)\times 1}{[0]} \\ \underset{l(2l)\times 1}{[r]} \end{array}\right)}$$

or more compactly,

$$\underset{l(5l+1)\times l(5l+1)}{J}\ \underset{l(5l+1)}{\bar{\theta}} = \underset{l(5l+1)\times 1}{j},$$

where

$$J = \begin{bmatrix} \underset{(3l+1)\times(5l+1)}{H} \\ \underset{l(2l)\times l(5l+1)}{R} \end{bmatrix} \text{ and } j = \begin{bmatrix} \underset{l(3l+1)\times 1}{h} \\ \underset{l(2l)\times 1}{r} \end{bmatrix}.$$

As long as J is invertible, the econometrician can recover the equivalent model $\bar{\theta}$ from the estimates of the reduced form (contained in H) and the restrictions, contained in (10).

In section 6 we conduct a set of experiments with our algorithm. In all of these experiments we assume that the DGP is determinate but in some of our examples, the equivalent model may have one or more degrees of indeterminacy. If the equivalent model is indeterminate then sunspot shocks may influence the solution and the matrix of impact effects for the non-fundamental shocks, $\bar{\Psi}_W^*$, may be non-zero.

The reduced form of the equivalent model might, in general, be hit by both fundamental and sunspot shocks. More generally, the matrix of impact effects of the shocks that hit the reduced form of the equivalent model might be different from the matrix of impact effects of the shocks that hit the reduced form of the DGP. In order for the reduced form of the equivalent model to be identical to the reduced form of the DGP it must, therefore, be driven by shocks with a different variance-covariance matrix from those of V_t. Our algorithm computes a variance covariance matrix of shocks to the equivalent model (we call this Ω) that makes the structural form of the DGP and the structural form of the equivalent model observationally equivalent.

We assume that the equivalent model is driven by a vector of serially uncorrelated shocks that we call

$$\begin{bmatrix} \underset{l\times 1}{\bar{V}_t} \\ \underset{k\times 1}{\bar{W}_t} \end{bmatrix}$$

where k is the dimension of indeterminacy of the equivalent model. We assume further that these shocks have the following properties,

$$E\begin{bmatrix}\bar{V}_t\\ \bar{W}_t\end{bmatrix}=0,\ E\left[\begin{pmatrix}\bar{V}_t\\ \bar{W}_t\end{pmatrix}\begin{pmatrix}\bar{V}_t\\ \bar{W}_t\end{pmatrix}'\right]=\Omega.$$

Our algorithm generates a matrix Ω and a set of impact matrices $\bar{\Psi}^*_V$, $\bar{\Psi}^*_W$, with the property that

$$\Psi^*_V I \Psi^{*\prime}_V=\begin{bmatrix}\bar{\Psi}^*_V & \bar{\Psi}^*_W\end{bmatrix}\Omega\begin{bmatrix}\bar{\Psi}^{*\prime}_V\\ \bar{\Psi}^{*\prime}_W\end{bmatrix}.$$

We next discuss the fact that reduced form of the equivalent model, generated by our algorithm, and the reduced form of the DGP may be different but equivalent representations of the same equations. Let Equation (11) represent the reduced form of the DGP

$$X_t=\Gamma^* X_{t-1}+C^*+\Psi^*_V V_t. \tag{11}$$

This representation will not be unique if, as is typically the case, Γ^* has one or more zero eigenvalues. Our Matlab code generates one representation of the reduced form of the DGP and a (possibly different) representation of the reduced form of the equivalent system that we write as follows;

$$X_t=\bar{\Gamma}^* X_{t-1}+\bar{C}^*+\bar{\Psi}^*_V \bar{V}_t+\bar{\Psi}^*_W \bar{W}_t. \tag{12}$$

We provide a separate algorithm that writes a matrix Γ^* in the form

$$\Gamma^*=QSZ',$$

where Q and Z are orthonormal and S is upper-triangular. The matrix Γ^* has dimension $(2l\times 2l)$. Suppose that n of its roots are non-zero and $2l-n$ roots are zero. Our algorithm chooses S such the $2l-n$ zero roots of Γ^* appear as diagonal elements on the lower right of the main diagonal. Using this decomposition we show how to rearrange equations (11) and (12) so that the first n columns of Γ^* and $\bar{\Gamma}^*$ are non-zero and the last $2l-n$ columns are all zeros. We write these rearranged equations as follows:

$$X_t = \Gamma_D^* X_{t-1} + C_D^* + \Psi_{Da}^* V_t + \Psi_{Db}^* V_{t-1}. \tag{13}$$

Equation (13) is the rearranged version of Equation (11) and Equation (14), given below

$$\begin{aligned} X_t = \overline{\Gamma}_D^* X_{t-1} + \bar{C}_D^* + \left[\begin{array}{cc} \bar{\Psi}_{DVa}^* & \bar{\Psi}_{DWa}^* \end{array} \right] \left[\begin{array}{c} \bar{V}_t \\ \bar{W}_t \end{array} \right] \\ + \left[\begin{array}{cc} \bar{\Psi}_{DVb}^* & \bar{\Psi}_{DWb}^* \end{array} \right] \left[\begin{array}{c} \bar{V}_{t-1} \\ \bar{W}_{t-1} \end{array} \right], \end{aligned} \tag{14}$$

is the rearranged version of Equation (12).

The rearranged equations may contain moving average errors even if the shocks to the original system are i.i.d. To accommodate this possibility we have introduced the notation Ψ_{Da}^*, Ψ_{Db}^* to refer to the impact matrices of fundamental shocks to Equation (13), and $\bar{\Psi}_{DVa}^*,\ \bar{\Psi}_{DWa}^*,\ \bar{\Psi}_{DVb}^*,\ \bar{\Psi}_{DWb}^*$ for the impact matrices of fundamental and sunspot shocks to Equation (14).

In Section 6 of the paper we use a second algorithm to establish that the equivalent system and the DGP are equivalent representations of the same reduced form. We check for each example, that (13) and (14) have the following properties

$$\Gamma_D^* = \overline{\Gamma}_D^*, \quad C_D^* = \bar{C}_D^*,$$

$$\Psi_{Da}^* I_l \Psi_{Da}^{*\prime} = \left[\begin{array}{cc} \bar{\Psi}_{DVa}^* & \bar{\Psi}_{DWa}^* \end{array} \right] \Omega \left[\begin{array}{cc} \bar{\Psi}_{DVa}^* & \bar{\Psi}_{DWa}^* \end{array} \right]',$$

and

$$\Psi_{Db}^* I_l \Psi_{Db}^{*\prime} = \left[\begin{array}{cc} \bar{\Psi}_{DVb}^* & \bar{\Psi}_{DWb}^* \end{array} \right] \Omega \left[\begin{array}{cc} \bar{\Psi}_{DVb}^* & \bar{\Psi}_{DWb}^* \end{array} \right]'.$$

These properties imply that the reduced forms of the two systems are identical when the DGP is driven by the shocks V_t with covariance matrix I_l and the equivalent system is driven by the shocks $\left[\bar{V}_t, \bar{W}_t\right]$ with covariance matrix Ω.

Table 1 summarizes our notation.

Table 1: Notational Conventions used in the Paper

	DGP	Equivalent Model
Structure	$\underset{l\times l}{A}, \underset{l\times l}{F}, \underset{l\times l}{B_1}, \underset{l\times l}{B_2}, \underset{l\times l}{\Psi_v}, \underset{l\times 1}{C}$	$\underset{l\times l}{\bar{A}}, \underset{l\times l}{\bar{F}}, \underset{l\times l}{\bar{B}_1}, \underset{l\times l}{\bar{B}_2}, \underset{l\times l}{\bar{\Psi}_v}, \underset{l\times 1}{\bar{C}}$
Companion form	$\underset{2l\times 2l}{\tilde{A}_0}, \underset{2l\times 2l}{\tilde{A}_1}, \underset{2l\times l}{\tilde{\Psi}_V}, \underset{2l\times l}{\tilde{\Psi}_W}, \underset{2l\times 1}{\tilde{C}}$	$\underset{2l\times 2l}{\bar{\tilde{A}}_0}, \underset{2l\times 2l}{\bar{\tilde{A}}_1}, \underset{2l\times l}{\bar{\tilde{\Psi}}_V}, \underset{2l\times l}{\bar{\tilde{\Psi}}_W}, \underset{2l\times 1}{\bar{\tilde{C}}}$
Reduced form	$\underset{2l\times 2l}{\Gamma^*}, \underset{2l\times l}{\Psi^*_V}, \underset{2l\times 1}{C^*}$	$\underset{2l\times 2l}{\bar{\Gamma}^*}, \underset{2l\times l}{\bar{\Psi}^*_V}, \underset{2l\times k}{\bar{\Psi}^*_W}, \underset{2l\times 1}{\bar{C}^*}$
Transformed R.f.	$\underset{2l\times 2l}{\Gamma^*_D}, \underset{2l\times l}{\Psi^*_{DVa}}, \underset{2l\times l}{\Psi^*_{DVb}}, \underset{2l\times 1}{C^*_D}$	$\underset{2l\times 2l}{\overline{\Gamma}^*_D}, \underset{2l\times l}{\bar{\Psi}^*_{DVa}}, \underset{2l\times l}{\bar{\Psi}^*_{DVb}}, \underset{2l\times k}{\bar{\Psi}^*_{DWa}}, \underset{2l\times k}{\bar{\Psi}^*_{DWb}}, \underset{2l\times 1}{\bar{C}^*_D}$
VCV Matrix	$\underset{l\times l}{V} \equiv I_l$	$\underset{(l+k)\times(l+k)}{\Omega}$

4 A One Equation Example

This section uses the one-equation example from our paper [3] to illustrate our algorithm in action. In this example the structural model is given by the equation

$$\begin{aligned} y_t &= \left(\frac{1}{\mu+\lambda}\right) E_t\left[y_{t+1}\right] + \left(\frac{\lambda\mu}{\lambda+\mu}\right) y_{t-1} + v_t, \qquad (15) \\ E_{t-1}\left[v_t v_s\right] &= \begin{cases} \sigma_v^2 & t = s \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

where $\mu > 1 > \lambda > 0$. We have parameterized the model in terms of λ and μ, the roots of the characteristic polynomial of (15) although one could clearly choose instead to write it in terms of the compound parameters $a = \left(\frac{1}{\mu+\lambda}\right)$ and $b = \left(\frac{\lambda\mu}{\lambda+\mu}\right)$.

The companion form of this one-equation system is

$$\overset{\tilde{A}_0}{\begin{bmatrix} 1 & -\left(\frac{1}{\mu+\lambda}\right) \\ 1 & 0 \end{bmatrix}} \overset{X_t}{\begin{bmatrix} y_t \\ E_t\left[y_{t+1}\right] \end{bmatrix}} = \overset{\tilde{A}_1}{\begin{bmatrix} \frac{\lambda\mu}{\lambda+\mu} & 0 \\ 0 & 1 \end{bmatrix}} \overset{X_{t-1}}{\begin{bmatrix} y_{t-1} \\ E_{t-1}\left[y_t\right] \end{bmatrix}} + \overset{\tilde{\Psi}_v}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} v_t + \overset{\tilde{\Psi}_w}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} w_t \tag{16}$$

$$E_{t-1}\left[w_t w_s\right] = \begin{cases} \sigma_w^2 & t = s \\ 0 & \text{otherwise} \end{cases},$$

where w_t is a non-fundamental error .

The reduced form is

$$\begin{bmatrix} y_t \\ E_t\left[y_{t+1}\right] \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ \lambda^2 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ E_{t-1}\left[y_t\right] \end{bmatrix} + \begin{bmatrix} \frac{(\lambda+\mu)}{\mu} \\ \frac{\lambda(\lambda+\mu)}{\theta} \end{bmatrix} v_t. \tag{17}$$

Our first two examples in Section 6 are elaborations of the idea that the parameter μ (the unstable root) cannot be recovered from equation (17) unless the econometrician has prior knowledge of the variance parameter σ_v^2. λ can be estimated from the dynamics of y_t but μ appears only in the variance of the residual where it cannot be separated from λ and σ_v^2. We will show that systems of equations can be constructed in which sets of parameters are not identified because the econometrician cannot obtain information on the magnitude of the unstable roots of the system.

The second two examples in Section 6 are based on the fact that one can find an indeterminate model that has an observationally equivalent reduced form to that of Equation (15). In the one-equation case, the alternative model is represented by the equation

$$y_t = \left(\frac{1}{\lambda}\right) E_t[y_{t+1}], \tag{18}$$

where $0 < \lambda < 1$. This system has the companion form

$$\overset{\bar{A}_0}{\begin{bmatrix} 1 & -\frac{1}{\lambda} \\ 1 & 0 \end{bmatrix}} \overset{X_t}{\begin{bmatrix} y_t \\ E_t\left[y_{t+1}\right] \end{bmatrix}} = \overset{\bar{A}_1}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \overset{X_{t-1}}{\begin{bmatrix} y_{t-1} \\ E_{t-1}\left[y_t\right] \end{bmatrix}} + \overset{\bar{\Psi}_w}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} w_t$$

and its reduced form is given by

$$\begin{bmatrix} y_t \\ E_t\left[y_{t+1}\right] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} y_{t-1} \\ E_{t-1}\left[y_t\right] \end{bmatrix} + \begin{bmatrix} 1 \\ \lambda \end{bmatrix} w_t. \tag{19}$$

Equations (17) and (19) look different because the former system is driven by a fundamental shock v_t and the latter by a non-fundamental shock, w_t. Furthermore, in Equation (17), y_t and $E_t\left[y_{t+1}\right]$ are functions of y_{t-1} and in Equation (19) they are functions of $E_{t-1}\left[y_t\right]$. Our paper [3] showed that Equation (19) can be arranged to give the following alternative (and completely equivalent) representation of the same system,

$$\begin{bmatrix} y_t \\ E_t\left[y_{t+1}\right] \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ \lambda^2 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ E_{t-1}\left[y_t\right] \end{bmatrix} + \begin{bmatrix} 1 \\ \lambda \end{bmatrix} w_t. \tag{20}$$

Equation (20) has the same dynamic structure as Equation (17) but it is driven by a sunspot shock w_t as opposed to the fundamental shock v_t. If however, $\sigma_w = \left(\mu / \left(\lambda + \mu\right)\right) \sigma_v$ then the reduced form of the (determinate) structural model (15) and the reduced form of the (indeterminate) equivalent model (18) are identical and the two structural models are observationally equivalent.

Examples 3 and 4 in Section 6 are based on an elaboration of this example. A key feature that makes these examples more complicated than Examples 1 and 2 is that the indeterminate system (and or the determinate system) needs to be rearranged so that the same lagged variables appear as right-hand-side variables in the reduced form.

5 A Summary of the New-Keynesian Model

In the remainder of this paper we deal explicitly with the following class of three-equation monetary models. We maintain the convention that endogenous variables (including expectations) appear on the left-hand-side of the equations and exogenous and predetermined variables appear on the right-hand-side. We let $Y_t = \{u_t, \pi_t, i_t\}$ where u_t is unemployment, π_t is the inflation rate and i_t is the nominal interest rate. By convention, u_t is the first element of Y_t, π_t is the second element and i_t is the third. We normalize the first equation by setting the coefficient on unemployment to unity, the second equation by setting the coefficient on inflation to unity and the third equation by setting the coefficient on the interest rate to unity. Using this labeling convention we refer to the first equation as the unemployment equation, the second as the inflation equation and the third as the interest-rate equation.[4]

The New-Keynesian model consists of three equations. The first is derived from the Euler Equation of an optimizing agent, which, in its simplest version, takes the form

$$U_C\left(C_t\right) = E_t\left[\beta U_C\left(C_{t+1}\right)\left(1+i_t\right)\frac{P_t}{P_{t+1}}\right]. \tag{21}$$

In this expression C_t is aggregate consumption, i_t is the nominal interest rate, P_t is the nominal price level, β is the discount factor of the representative agent, E_t is the conditional expectations operator and U_C represents the marginal utility of consumption. To allow for the fact that the data has a richer dynamic pattern than that of simple models, some authors also allow for habit formation in preferences.[5] A simple form of habit formation would

[4] Much of the literature uses the output gap in place of unemployment. Since our own empirical work has used the unemployment rate, we chose unemployment as the scale variable in our work on identification.

[5] Fuhrer ([8]) is an example.

permit lagged consumption to enter utility; thus Equation (21) would take the form

$$U_C\left(C_t, C_{t-1}\right) = E_t\left[\beta U_C\left(C_{t+1}, C_t\right)\left(1+i_t\right)\frac{P_t}{P_{t+1}}\right]. \tag{22}$$

When equation (22) is linearized around a balanced growth path it leads to the log-linear equation

$$a_1 c_t + a_2 c_{t-1} = a_1 E_t\left[c_{t+1}\right] + a_2 c_t + a_3\left(i_t - \pi_{t+1}\right) + c_1 + v_t,$$

where lower case c_t is the log of consumption, π_{t+1} is the log of inflation, i_t is the nominal interest rate, v_t is a preference shock and c_1 is a constant.

New Keynesian authors often work with three-equation models in which consumption is replaced by output and v_t is interpreted in part as a government expenditure shock.[6] In our work, we will go one step further than this and replace c_t by the negative of unemployment. Implicitly, we are appealing to Okun's law which is a relationship between unemployment and the output gap. Using these arguments, our first equation takes the form,

$$u_t + a_{13} i_t + f_{11} E_t\left[u_{t+1}\right] + f_{12} E_t\left[\pi_{t+1}\right] = b_{11} u_{t-1} + c_1 + v_{1t}. \tag{23}$$

We use the symbols a_{ij}, f_{ij} and b_{ij} to represent arbitrary elements of the matrices A, F and B, where i indexes equation and j indexes variable. Hence a_{13} is the coefficient on the interest rate in the unemployment equation, b_{11} and f_{11} are the coefficients on lagged unemployment and expected future unemployment and f_{12} is the coefficient on expected future inflation. Since the coefficients on unemployment are derived from combining the linearized version of Equation (22) with Okun's law, economic theory imposes the equality restrictions,

$$\begin{aligned} 1 + f_{11} &= b_{11}, & (24)\\ f_{12} + a_{13} &= 0. & (25)\end{aligned}$$

[6] Examples include Rotemburg and Woodford [24], or McCallum and Nelson [20] who refer to an equation with this structure as an optimizing based IS-curve.

The first of these restrictions follows from the fact that the linearization coefficients on c_t and c_{t-1} on the left hand side of equation (22) are the same as those premultiplying $E_t[c_{t+1}]$ and c_t on the right hand side. The second restriction follows from the assumption that the real interest rate, not the nominal interest rate, enters the IS curve.

The second equation of the New-Keynesian model is the Phillips Curve also known as the New-Keynesian aggregate supply curve, (AS-curve). This equation takes the form;

$$\pi_t + a_{21}u_t + f_{22}E_t[\pi_{t+1}] = c_2 + v_{2t}. \tag{26}$$

Economic theory predicts that a_{21} should be positive, reflecting the fact that high unemployment puts downward pressure on inflation. The coefficient f_{22} represents the effect of expected future inflation on current price setting decisions. According to simple versions of the New Keynesian model this parameter should be equal to the negative of the rate of time preference of the representative agent. We will set it equal to -0.97, i.e. $f_{22} = -0.97$ in our simulations. This is consistent with a steady state real rate of interest of 3%.

To close the New Keynesian model we assume that policy is generated by a reaction function of the form

$$\begin{aligned} i_t + a_{31}u_t + a_{32}\pi_t + f_{31}E_t[u_{t+1}] + f_{32}E_t[\pi_{t+1}] + f_{33}E_t[i_{t+1}] \\ = b_{31}u_{t-1} + b_{32}\pi_{t-1} + b_{33}i_{t-1} + c_3 + v_{3t}. \end{aligned} \tag{27}$$

This is a very general form of the policy rule and in practice we will consider simpler examples in which the Central Bank responds either to lagged inflation or to expected future inflation. We will also allow for slow adjustment of the policy maker to target by including the lagged interest rate in the policy rule. One example of a simple rule that has been widely studied in the literature is given by the equation:

$$i_t + f_{32} E_t [\pi_{t+1}] = b_{33} i_{t-1} + c_3 + v_{3t}. \qquad (28)$$

In the following section we will use this three-equation example to illustrate a number of points connected with identification in linear rational expectations models. We have chosen this particular model for ease of exposition and it should be apparent that the same points that we make will apply to models in which unemployment is replaced by the output-gap, or when the system is expanded to include a wage sector and a measure of marginal cost as in the work of Gali and Gertler [11] or Argia Sbordone [27].

6 Experiments with the New Keynesian Model

In this section we use our Matlab code to illustrate four points connected with identification in rational expectations models. We begin by conducting an experiment in which we point out that the policy rule of a monetary model is not generally identified when one recognizes that there is no compelling reason to exclude backward, current or forward looking variables.

Our second experiment goes on to show that lack of identification extends to the other equations of a New-Keynesian model. Specifically, we show that it is not possible to identify the Phillips curve without an arbitrary decision about which variables enter the IS curve and the policy rule. This second experiment throws doubt on the arguments of Lindé [15] and Rudd and Whelan [23] who argue that backward looking Phillips curve perform better than forward looking Phillips curves at explaining data. Our experiment suggests that their results hinge on arbitrary exclusion restrictions made somewhere in the system of equations in order to achieve identification of the Phillips curve.

In experiments 3 and 4 we move from systems in which equilibria are determinate to systems in which one or more identification schemes are associated with indeterminate equilibria. These experiments yield results that are successively more damaging to a research program that hopes to uncover a unique representation of the data from a careful combination of microfoundations and econometrics.

6.1 Experiment 1: Policy Rules are Not Identified

We begin with the issue of identification of the policy rule. A number of authors, (an example is the work of Clarida, Gali and Gertler [4]), have used instrumental variables to estimate policy rules in new Keynesian models. Our first experiment begins with a model of the form

$$\begin{aligned} u_t &= E_t\left[u_{t+1}\right] + 0.05\left(i_t - E_t\left[\pi_{t+1}\right]\right) - 0.0015 + v_{1t}, && (29) \\ \pi_t &= 0.97E_t\left[\pi_{t+1}\right] - 0.5u_t + 0.0256 + v_{2t}, && (30) \\ i_t &= 1.1E_t\left[\pi_{t+1}\right] + 0.028 + v_{3t}. && (31) \end{aligned}$$

The variable u_t is the logarithm of unemployment, π_t is the inflation rate and i_t is the nominal interest rate. Equation (29) is an optimizing IS curve, Equation (30) is a New-Keynesian aggregate supply curve and (31) is a policy rule. The constants are chosen such that the model has a steady state unemployment rate of 5%, a real interest rate of 3% and a steady state inflation rate of 2%. This model has the following reduced form

$$\begin{bmatrix} u_t \\ \pi_t \\ i_t \end{bmatrix} = \begin{bmatrix} 0.05 \\ 0.02 \\ 0.05 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0.05 \\ -0.5 & 1 & -0.25 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{1t} \\ v_{2t} \\ v_{3t} \end{bmatrix}, \qquad (32)$$

in which there are no intrinsic dynamics. It is well known that this model has a unique determinate equilibrium as long as the government follows a policy in which the coefficient of inflation in the policy rule is greater than 1; following the work of Eric Leeper[14], a policy with this property is called *active*. But all policies of the form

$$i_t = -f_{32} E_t \left[\pi_{t+1}\right] + c_3 + v_{3t},$$

for which

$$|f_{32}| > 1$$

lead to exactly the same reduced form as Equation (32), as long as c_3 and f_{32} are chosen to preserve the same steady state interest rate. More generally, any policy the class

$$i_t = -a_{31} u_t - a_{32} \pi_t - f_{31} E_t \left[u_{t+1}\right] - f_{32} E_t \left[\pi_{t+1}\right] + c_3 \tag{33}$$

will lead to the same reduced form, i.e.

$$\begin{bmatrix} u_t \\ \pi_t \\ i_t \end{bmatrix} = \begin{bmatrix} 0.05 \\ 0.02 \\ 0.05 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0.05 \\ -0.5 & 1 & -0.25 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{1t} \\ v_{2t} \\ v_{3t} \end{bmatrix},$$

provided the policy preserves the same degree of determinacy as rule (31).

6.2 Experiment 2: We Can't Tell if the Phillips Curve is Backward or Forward Looking

There is a debate in the literature over the exact form of the aggregate supply curve. In simple New Keynesian theories the aggregate supply curve is represented by a forward looking Phillips curve, but Jeffrey Fuhrer [7] has argued that the data is better fit by a model with a backward looking inflation term. The following example illustrates that this point may not be decidable purely from the data since the supply curve may not be identified.

Consider the model

$$u_t = E_t[u_{t+1}] + 0.05(i_t - E_t[\pi_{t+1}]) - 0.0015 + v_{1t}, \tag{34}$$

$$\pi_t = 0.97E_t[\pi_{t+1}] - 0.5u_t + 0.0256 + v_{2t}, \tag{35}$$

$$i_t = 1.1\pi_{t-1} + 0.028 + v_{3t}, \tag{36}$$

in which the supply curve is forward looking but the central bank responds to lagged inflation. Our second experiment uses Equations (34)–(36) as the DGP and correctly identifies the IS curve and the policy rule. However, the econometrician incorrectly identifies a hybrid aggregate supply curve of the form

$$\pi_t = -f_{22}E_t[\pi_{t+1}] + b_{22}\pi_{t-1} - a_{21}u_t + c_2 + v_{2t}. \tag{37}$$

The reduced form of the DGP is the model,

$$X_t = C^* + \Gamma^* X_{t-1} + \Psi_V^* V_t,$$

where

$$C^* = \begin{bmatrix} 0.0489 \\ 0.0205 \\ 0.0280 \\ 0.0500 \\ 0.0200 \\ 0.0506 \end{bmatrix}, \quad \Gamma^* = \begin{bmatrix} 0 & 0 & 0.0536 & 0 & 0 & 0 \\ 0 & 0 & -0.0261 & 0 & 0 & 0 \\ 0 & 0 & 1.1000 & 0 & 0 & 0 \\ 0 & 0 & -0.0014 & 0 & 0 & 0 \\ 0 & 0 & 0.0007 & 0 & 0 & 0 \\ 0 & 0 & -0.0287 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\Psi^* = \begin{bmatrix} 0.9739 & 0.0521 & 0.0487 \\ -0.4749 & 0.9499 & -0.0237 \\ 0 & 0 & 1 \\ -0.0254 & 0.0509 & -0.0013 \\ 0.0124 & -0.0248 & 0.0006 \\ -0.5224 & 1.0449 & -0.0261 \end{bmatrix}.$$

Our algorithm reveals a large class of hybrid models, all of which have the same reduced form as the DGP. The following table presents five examples of models in this class.

Coefficient	f_{22}	b_{22}	a_{21}	c_2
	-0.97	0	-0.5	0.0256
	-0.97	0.25	-5.1671	0.2540
	-0.97	0.5	-9.8342	0.4823
	-0.97	0.75	-14.5013	0.7107
	-0.97	1.0	-19.1683	0.9390

Table 2

In experiment 2, the econometrician correctly identifies the coefficient on expected future inflation, equal to 0.97, but he incorrectly estimates a non-zero parameter on lagged inflation. Table 2 shows how the constant and the coefficient on unemployment are adjusted to compensate for different alternative values of the parameter b_{22}, (the coefficient on lagged inflation). All of the models recorded in Table 2 deliver the same reduced form model as the DGP.

6.3 Experiment 3: We Can't Distinguish Alternative Theories of Aggregate Supply

A number of authors have taken up the issue of optimal policy in the new-Keynesian model. Michael Woodford [28] has argued that the central bank should strive to implement a policy that leads to a unique determinate rational expectations equilibrium since, if policy admits the possibility of indeterminacy, non-fundamental shocks may contribute to the variance of inflation and unemployment. This consideration suggests that a policy maker that dislikes variance should pick a policy rule that leads to a determinate equilibrium.

In the class of new-Keynesian models represented by equations (29), (30) and (31), equilibrium is determinate if $f_{32} < -1$ and indeterminate if $0 > f_{32} > -1$. In the former case, the central bank increases the nominal interest rate by more than one-for-one if it expects additional future inflation; a policy with this property is said to be *active.* In the latter case the central bank increases the interest rate by less than one-for-one if it expects additional inflation and in this case the policy is said to be *passive.*

Benhabib and Farmer [1] have argued that prices are slow to adjust, not because of barriers or frictions that prevent agents from adjusting prices, but because equilibria are typically indeterminate. The Benhabib-Farmer model has the same IS curve as the new-Keynesian model, but a different supply curve that takes the form

$$u_t + a_{23} i_t = b_{21} u_{t-1} + c_2 + v_{2t}. \tag{38}$$

In this model, the parameter a_{23} is negative. The mechanism by which the nominal interest rate influences aggregate supply operates through liquidity effects in production; a lower nominal interest rate causes agents to hold more real balances and this increases economic activity. The parameter b_{21} allows the effects of high interest rate policies to be distributed through time. In a simple form of the Benhabib-Farmer model, with the additional simplifications of no habit formation, no lagged adjustment of the policy rule,no lagged adjustment of the supply curve and a policy rule of the form

$$i_t = -a_{32} \pi_t + c_3 + v_{3t} \tag{39}$$

equilibrium is determinate if $0 > a_{32} > -1$ and indeterminate if $a_{32} < -1$ This is exactly the opposite situation of the new-Keynesian model. In the Benhabib-Farmer model, active policy implies *indeterminacy* and passive policy implies determinacy.

In our third experiment we chose the DGP to consist of the following three equations

$$\begin{aligned}
u_t &= 0.5E_t\left[u_{t+1}\right] + 0.5u_{t-1} + 0.05\left(i_t - E_t\left[\pi_{t+1}\right]\right) - 0.0015 + v_{1t}, && (40)\\
\pi_t &= 0.97E_t\left[\pi_{t+1}\right] - 0.5u_t + 0.0256 + v_{2t}, && (41)\\
i_t &= 1.1E_t\left[\pi_{t+1}\right] - 0.012 + 0.8i_{t-1} + v_{3t}. && (42)
\end{aligned}$$

The IS curve contains lagged unemployment, reflecting habit formation in preferences, and the policy rule is forward looking with a lagged interest coefficient reflecting a theory of partial adjustment of the central bank towards its target. The AS curve is a standard forward looking new-Keynesian Phillips curve. The reduced form of the DGP for this model is

$$X_t = C^* + \Gamma^* X_{t-1} + \Psi_V^* V_t, \tag{43}$$

where

$$C^* = \begin{bmatrix} 0.0070 \\ 0.0766 \\ 0.0498 \\ 0.0176 \\ 0.0562 \\ 0.0725 \end{bmatrix}, \; \Gamma^* = \begin{bmatrix} 0.7520 & 0 & 0.1080 & 0 & 0 & 0 \\ -0.8393 & 0 & -0.2921 & 0 & 0 & 0 \\ -0.5254 & 0 & 0.5300 & 0 & 0 & 0 \\ 0.5087 & 0 & 0.1384 & 0 & 0 & 0 \\ -0.4777 & 0 & -0.2455 & 0 & 0 & 0 \\ -0.6736 & 0 & 0.2242 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\Psi_V^* = \begin{bmatrix} 1.5039 & 0 & 0.1350 \\ -1.6786 & 1 & -0.3651 \\ -1.0509 & 0 & 0.6625 \\ 1.0174 & 0 & 0.1731 \\ -0.9553 & 0 & -0.3068 \\ -1.3472 & 0 & 0.2802 \end{bmatrix}.$$

We assume that the econometrician correctly identifies the IS curve but he assumes incorrectly that aggregate supply is generated by the Benhabib Farmer model. Further, the econometrician incorrectly assumes that policy responds to current inflation, rather than to expected future inflation. The econometrician estimates the correct IS curve, but a supply curve and a policy rule of the form

$$\begin{aligned} u_t &= 0.2038 i_t + 0.859 u_{t-1} - 0.0031 + v_{2t}, & (44) \\ i_t &= 0.626 \pi_t + 0.7129 i_{t-1} + 0.0018 + v_{3t}. & (45) \end{aligned}$$

The reduced form of the Benhabib-Farmer equivalent model is given by the equation

$$X_t = \bar{C}^* + \bar{\Gamma}^* X_{t-1} + \bar{\Psi}_V^* V_t + \bar{\Psi}_W^* W_t, \tag{46}$$

where the matrices $\bar{C}^*$ and $\bar{\Gamma}^*$ are given by the expressions,

$$\bar{C}^* = \begin{bmatrix} -0.0028 \\ -0.0006 \\ 0.0015 \\ 0.0050 \\ 0.0785 \\ 0.0520 \end{bmatrix}, \ \bar{\Gamma}^* = \begin{bmatrix} 0.8491 & 0 & 0.10320 & 0.01160 & 0.0906 & 0.0567 \\ -0.0778 & 0 & -0.3301 & 0.0906 & 0.7101 & 0.4446 \\ -0.0487 & 0 & 0.5062 & 0.0567 & 0.4446 & 0.2783 \\ 0.6332 & 0 & 0.13220 & 0.01480. & 0.1161 & 0.0727 \\ -0.6984 & 0 & -0.2344 & -0.0263 & -0.2059 & -0.1289 \\ -0.4720 & 0 & 0.2141 & 0.0240 & 0.1880 & 0.1177 \end{bmatrix}$$

the impact matrix of fundamental and non-fundamental shocks is

$$\begin{bmatrix} \bar{\Psi}_V^* & \bar{\Psi}_W^* \end{bmatrix} = \begin{bmatrix} 1.6982 & -0.0906 & 0.1289 & 0.1075 \\ -0.1556 & 0.2899 & -0.4126 & 0.8427 \\ -0.0974 & -0.4446 & 0.6328 & 0.5275 \\ 1.2665 & -0.1161 & 0.1653 & 0.1378 \\ -1.3969 & 0.2059 & -0.2930 & -0.2443 \\ -0.9439 & -0.1880 & 0.2676 & 0.2231 \end{bmatrix},$$

and their covariance matrix Ω is

$$\Omega = \begin{bmatrix} 1 & 0 & 0 & -1.8073 \\ 0 & 3.0260 & 0 & 0.7625 \\ 0 & 0 & 0 & 0 \\ -1.8073 & 0.7625 & 0 & 3.9796 \end{bmatrix}. \tag{47}$$

where

$$\Psi_V^* \Psi_V^{*\prime} = \begin{bmatrix} \bar{\Psi}_V^* & \bar{\Psi}_W^* \end{bmatrix} \Omega \begin{bmatrix} \bar{\Psi}_V^* & \bar{\Psi}_W^* \end{bmatrix}' = \begin{bmatrix} 2.2800 & -2.5738 & -1.4910 \\ -2.5738 & 3.9511 & 1.5221 \\ -1.4910 & 1.5221 & 1.5432 \end{bmatrix}.$$

This example is unlike those from experiments 1 and 2 because the equivalent system is indeterminate. In experiment 3 our Matlab algorithm produces two equivalent reduced forms that use different right-hand-side variables. This equivalence is possible because, as in the one-equation example that we described in Section 4, the matrices Γ^* and $\bar{\Gamma}^*$ have one or more zero roots.

To check the equivalence of the two representations, we wrote a second algorithm that rewrites a first order $l \times 1$ matrix difference equation with n non-zero roots so that the first n variables of the system all appear as lagged variables on the right-hand-side. When we ran the reduced forms from experiment 3 through this algorithm we were able to represent the DGP in the form

$$X_t = \Gamma_D^* X_{t-1} + C_D^* + \Psi_{Da}^* V_t + \Psi_{Db}^* V_{t-1} \tag{48}$$

and the equivalent model as

$$X_t = \bar{\Gamma}_D^* X_{t-1} + \bar{C}_D^* + \begin{bmatrix} \bar{\Psi}_{DVa}^* & \bar{\Psi}_{DWa}^* \end{bmatrix} \begin{bmatrix} \bar{V}_t \\ \bar{W}_t \end{bmatrix} + \begin{bmatrix} \bar{\Psi}_{DVb}^* & \bar{\Psi}_{DWb}^* \end{bmatrix} \begin{bmatrix} \bar{V}_{t-1} \\ \bar{W}_{t-1} \end{bmatrix} \tag{49}$$

where

$$\begin{aligned}
\Gamma_D^* &= \overline{\Gamma}_D^*, \quad C_D^* = \bar{C}_D^*, \\
\Psi_{Da}^* I_l \Psi_{Da}^{*\prime} &= \begin{bmatrix} \bar{\Psi}_{DVa}^* & \bar{\Psi}_{DWa}^* \end{bmatrix} \Omega \begin{bmatrix} \bar{\Psi}_{DVa}^* & \bar{\Psi}_{DWa}^* \end{bmatrix}' \\
\Psi_{Db}^* I_l \Psi_{Db}^{*\prime} &= \begin{bmatrix} \bar{\Psi}_{DVb}^* & \bar{\Psi}_{DWb}^* \end{bmatrix} \Omega \begin{bmatrix} \bar{\Psi}_{DVb}^* & \bar{\Psi}_{DWb}^* \end{bmatrix}'.
\end{aligned}$$

Models (48) and (49) are observationally equivalent even though model (48) describes a determinate new-Keynesian DGP and model (49) is an indeterminate Benhabib-Farmer model in which the covariance matrix includes a sunspot. The variance covariance matrix of the driving process to the new-Keynesian model is diagonal and consists of three fundamental shocks of equal variance. The variance covariance matrix of the shocks to the Benhabib-Farmer model, given by Equation (47), has rank 3. Only two of the three fundamental shocks have non-zero variances. The third fundamental shock is replaced by a sunspot variable W_t that is correlated with the fundamentals.

6.4 Experiment 4: The Importance of Transparency

Experiments 1 through 3 highlight a point we made in our earlier paper [2]; it is difficult or impossible to credibly identify the monetary transmission process using data from a single sample in which all the parameters of the model remain invariant. The message we take away from these examples is that natural experiments of the kind we exploit in our paper [2] are essential if an econometrician is to be able to identify the structure of the economy. If the experiments do not occur "naturally" as a consequence of changes in political regimes then, in theory, it would make sense for a benevolent policy maker to engage in deliberate experimentation to learn about the nature of the private sector.

Our fourth and final experiment is designed to illustrate that experimentation on its own is not enough to tell whether a given policy rule led to a determinate or an indeterminate outcome; in that theoretical environment the policy maker must also be transparent in the following sense. We distinguish between the policy maker who conducts an experiment by changing the policy rule, and the econometrician who observes the outcome of the experiment. We assume that for a fixed period of time the policy maker follows policy rule A and that at a given date, announced in advance, he switches to policy rule B. The policy maker and the econometrician both observe the outcome of this experiment and, using the methods discussed in [2], they are each able to identify a subset of the equations that describe the structure of the private sector. Typically, however, the policy maker and the econometrician will have different information sets since the policy maker knows the rules that were followed during the two periods of the policy experiment but the econometrician may not.

In Experiment 4 we assume that the DGP is given by the same process as in Experiment 3, that is;

$$\begin{aligned} u_t &= 0.5E_t[u_{t+1}] + 0.5u_{t-1} + 0.05(i_t - E_t[\pi_{t+1}]) - 0.0015 + v_{1t}, && (50) \\ \pi_t &= 0.97E_t[\pi_{t+1}] - 0.5u_t + 0.0256 + v_{2t}, && (51) \\ i_t &= 1.1E_t[\pi_{t+1}] - 0.012 + 0.8i_{t-1} + v_{3t}. && (52) \end{aligned}$$

We assume further that the econometrician knows that Equations (50) and (51) are the true equations that characterize the private sector of the economy.[7] However, the econometrician does not know the policy rule that was followed by the central bank and he assumes, incorrectly, that the lagged

[7] By observing a natural experiment, the econometrician can learn a subset of the private sector structural equations; in Beyer Farmer [2] we call this subset the recoverable structure. In contrast, we assume in this section that the econometrician knows the structural form; this is more information than he could hope to learn through experiment.

interest rate does not enter this equation. The econometrician estimates instead, a policy rule of the form;

$$i_t = -4.6383u_t - 3.5295\pi_t + 0.3525 + v_{3t}. \tag{53}$$

The DGP leads to a reduced form that can be represented as

$$X_t = \Gamma^*_D X_{t-1} + C^*_D + \Psi^*_{Da} V_t + \Psi^*_{Db} V_{t-1} \tag{54}$$

and the equivalent model in which the econometrician misidentifies the policy rule has the representation,

$$X_t = \overline{\Gamma}^*_D X_{t-1} + \bar{C}^*_D + \begin{bmatrix} \bar{\Psi}^*_{DVa} & \bar{\Psi}^*_{DWa} \end{bmatrix} \begin{bmatrix} \bar{V}_t \\ \bar{W}_t \end{bmatrix} + \begin{bmatrix} \bar{\Psi}^*_{DVb} & \bar{\Psi}^*_{DWb} \end{bmatrix} \begin{bmatrix} \bar{V}_{t-1} \\ \bar{W}_{t-1} \end{bmatrix}, \tag{55}$$

where

$$\begin{aligned} \Gamma^*_D &= \overline{\Gamma}^*_D, \quad C^*_D = \bar{C}^*_D, \\ \Psi^*_{Da} I_l \Psi^{*\prime}_{Da} &= \begin{bmatrix} \bar{\Psi}^*_{DVa} & \bar{\Psi}^*_{DWa} \end{bmatrix} \Omega \begin{bmatrix} \bar{\Psi}^*_{DVa} & \bar{\Psi}^*_{DWa} \end{bmatrix}' \\ \Psi^*_{Db} I_l \Psi^{*\prime}_{Db} &= \begin{bmatrix} \bar{\Psi}^*_{DVb} & \bar{\Psi}^*_{DWb} \end{bmatrix} \Omega \begin{bmatrix} \bar{\Psi}^*_{DVb} & \bar{\Psi}^*_{DWb} \end{bmatrix}' \end{aligned}$$

and the equivalent model is driven by two fundamental shocks and one sunspot, just as with Example 3. This Experiment is discouraging for the ability of careful econometrics to distinguish good and bad policies since it suggests that even if the econometrician knows the true structure of the private sector, he cannot tell whether the policy maker followed a rule that led to a determinate or an indeterminate equilibrium.[8]

However, since the recoverable form and the structural equations are equivalent ways of modeling the private sector response to any given policy rule, it makes no difference to our argument if we give the econometrician this additional knowledge about the private sector.

[8]This is in contrast to Lubik and Schorfheide [17], who claim to be able to distinguish determinate from indeterminate equilibria. Their procedure relies on a priori restrictions to the lag length of both the structure and the policy rule. The fact that restrictions of this kind are important was first pointed out by Pesaran [22].

7 Conclusions

What did we learn from this paper? Let's imagine a discussion where a policy maker turns to a new-Keynesian economist. The new-Keynesian economist gives the following advice.

"Mr. Chairman, I have estimated the response of the economy to changes in the interest rate and I counsel that policy should be active. You would be wise to increase the fed funds rate by more than one-for-one in response to an increase in expected inflation since a rule of this kind will minimize the danger of irrational exuberance as a separate impulse to economic fluctuations."

Experiment 3 suggests that this advice is overconfident at best. A policy advisor who had read the Benhabib-Farmer [1] paper might offer these alternative words of wisdom.

"Mr. Chairman, my esteemed colleague is clearly mistaken since he is working with a wrong and outdated theory of aggregate supply. I have checked his conclusions and I find that the policy rule he suggests will have exactly the opposite effects from those that he predicts. You would be wise to move cautiously by increasing the fed funds rate less than one for one in response to an increase in expected inflation."

How could the policy maker decide which of these two economists is correct? If the economy were a laboratory and the policy maker were a scientist, one could imagine the following experiment. The open market committee could fix on a rule for a given period of, say, five years. We imagine that the parameters of this rule would be known and announced in advance. At the end of this five year period the open market committee would change its policy to a different preannounced rule and this second rule would be followed for an equal period of time. An experiment of this kind would allow an impartial econometrician to use the break in policy to identify the structural equations of the private sector by following the approach discussed in [2].

Transparency is crucial for this policy to work. Rules must be announced to the public, but also, the econometrician must be kept aware of their details. It is only with this information in hand that the econometrician can completely identify the structural parameters of the economy and thereby give advice as to the welfare properties of alternative rules.

The economy, however, is not a laboratory. One would hope that designing monetary policy in the spirit of the controlled experiment discussed in this paper would contribute to social welfare - but it would induce also costs. In order to balance the costs and benefits one would need to find a measure for an optimal degree of "experimentation". This might be an interesting avenue for future research.

References

[1] Benhabib, Jess and Roger E. A. Farmer, (2000). "The Monetary Transmission Mechanism" *Review of Economic Dynamics* Vol. 3,523–550.

[2] Beyer, Andreas and Roger E. A. Farmer (2003a). "Identifying the Monetary Transmission Mechanism Using Structural Breaks", European Central Bank working paper #275.

[3] ______________________________________(2003b). "On the Indeterminacy of Determinacy and Indeterminacy", European Central Bank working paper #277.

[4] Clarida, R., Gali, J. and M. Gertler (2000), "Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory", *Quarterly Journal of Economics*, 115 (1), pp.147-180.

[5] Cole, Harold L. and Lee Ohanian, (1999). "Aggregate Returns to Scale: Why Measurement is Imprecise" *Federal Reserve Bank of Minneapolis Quarterly Review* Vol. 23, No. 3, Summer, 19–28.

[6] Farmer, Roger E. A. and Jang-Ting Guo, (1995). "The Econometrics of Indeterminacy" *Carnegie Rochester Conference Series on Public Policy,* 42, June.

[7] Fuhrer, Jeffrey C., (1997). "The (Un)Importance of Forward-Looking Behavior in Price Specifications," *Journal of Money Credit and Banking* 29 (3) 338-350.

[8] ________________, (2000). "Habit Formation in Consumption and its Implications for Monetary Policy Models," *American Economic Review*, 90, no. 3, pp. 367–90.

[9] Fuhrer, Jeffrey C., and George R. Moore (1995). "Inflation Persistence", *Quarterly Journal of Economics*, 440, pp.127-159.

[10] Fuhrer, Jeffrey C. and Glenn D. Rudebusch (2003). "Estimating the Euler equation for output," Working Papers in Applied Economic Theory 2002-12, Federal Reserve Bank of San Francisco

[11] Gali, Jordi and Mark Gertler, (1999) "Inflation Dynamics: A Structural Econometric Analysis", *Journal of Monetary Economics.*

[12] Gordon, Robert J.. (1997). "The Time-Varying NAIRU and its Implications for Economic Policy," *Journal of Economic Perspectives* 11, Winter, 11–32.

[13] Kamihigashi, Takashi, (1996). "Real Business Cycles and Sunspot Fluctuations are Observationally Equivalent" *Journal of Monetary Economics* 37, February, 105–17.

[14] Leeper, Eric M. (1991). "Equilibria under 'Active' and 'Passive' Monetary and Fiscal Policies", *Journal of Monetary Economics*, 27, pp 129–147.

[15] Lindé, Jesper, (2001). "The Empirical Relevance of Simple Forward- and Backward-looking Models: A View from a Dynamic General Equilibrium Model", Sveriges Riksbank (Central Bank of Sweden) Working paper #130.

[16] ____________________ "Estimating New-Keynesian Phillips Curves: A Full Information Maximum Likelihood Approach", Sveriges Riksbank (Central Bank of Sweden) Working paper #129.

[17] Lubik, Thomas A. and Frank Schorfheide. (2002a). "Computing Sunspot Equilibria in Linear Rational Expectations Models," mimeo, University of Pennsylvania.

[18] ____________________________________ (2002b). "Testing for Indeterminacy: An Application to U.S. Monetary Policy," *American Economic Review* , forthcoming.

[19] Mavroeidis, Sophocles, (2002). "Identification and Mis-Specification Issues in Forward Looking Models", Department of Quantitative Economics, University of Amsterdam, Discussion paper 2002/21.

[20] McCallum, B.T. and Nelson, E. (1999). "An Optimizing IS-LM Specification for Monetary Policy and Business Cycle Analysis," *Journal of Money, Credit, and Banking* 31 (August 1999, Part 1), 296-316 .

[21] Nason, James M. and Gregor W. Smith, (2003). "Identifying the New Keynesian Phillips Curve". Mimeo, Federal Reserve Bank of Atlanta.

[22] Pesaran, Hashem, (1987). *The Limits to Rational Expectations* Basil Blackwell, Oxford.

[23] Rudd, Jeremy and Karl Whelan, (2001). "New Tests of the New-Keynesian Phillips Curve", Federal Reserve Board, Washington D.C. Finance and Economics Discussion Series 2001-30.

[24] Rotemberg, J.J and M. Woodford (1998), "An optimization-based econometric framework for the evaluation of monetary policy: expanded version", NBER Technical Working Paper 233.

[25] Sims, Christopher A., (1980) "Macroeconomics and Reality," Econometrica, 48, 1, pp. 1-48.

[26] Sims, Christopher, A. (1996). "Solving Linear Rational Expectations Models". Princeton University mimeo.

[27] Sbordone, Argia, (2001). "An Optimizing Model of U.S. Wage and Price Dynamics". Mimeo, Rutgers University.

[28] Woodford, Michael, (2003). *Interest and Prices* Princeton University Press.