Time Consistent Control in Non-Linear Models

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January 2005

Abstract

This paper shows how to use optimal control theory to derive time-consistent optimal government policies in nonlinear dynamic general equilibrium models. It extends the insight of Cohen and Michel (1988), who showed that in *linear* models time-consistent policies can be found by imposing a linear relationship between predetermined state variables and the costate variables from private agents' maximization problems. We use an analogous procedure based on the Den Haan and Marcet (1990) technique of parameterized expectations, which replaces nonlinear functions of expected future costates by flexible functions of current states. This leads to a *nonlinear* relationship between current state and costate variables, which is verified in equilibrium to an arbitrarily close degree of approximation. The optimal control problem of the government is recursive, unlike the Ramsey (1927) problem which is common in the optimal taxation literature. We use a model of public investment to illustrate the technique.

Keywords: Optimal government policy; Time consistent control
JEL classification codes: E61, E62, C63

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1 Introduction

One of the appealing features of solving Ramsey (1927) problems to derive optimal second-best government policies in dynamic general equilibrium models is their relative analytical tractability. It is often possible to use the so-called *primal approach*, in which private agents' first order conditions and budget constraints are combined to derive an *implementability constraint*,[1] in which prices and policy variables are substituted out of the problem. The choice variables of the optimal policy problem are the variables of the optimal intertemporal allocations. The values of prices and policy variables that support the optimal allocations can be derived once the allocations themselves are known. Using the primal approach leads to equations in which expected future allocations have an influence on agents' current behavior. Therefore, optimal policies derived in this manner are typically *time-inconsistent*. The government must be able to credibly commit to its announced policies. Otherwise, it will optimally revise them as time goes by, in which case its announced policies will not be believed by private agents.

It is often interesting to compare the optimal allocations under credible precommitment by the government to optimal allocations where this precommitment is not possible, possibly for institutional or political reasons. Unfortunately, technical difficulties often limit our ability to do this. One approach to deriving optimal time-consistent policies involves linearizing the laws of

[1] See Chari and Kehoe (1998) for a detailed explanation.

motion of the economy and by using quadratic approximations to agents' preferences.[2] This approach may be less than satisfactory in the presence of important nonlinearities. It may also be inappropriate for analysing transition paths to steady states which are sufficiently far from initial conditions that linear approximations break down. It would be useful to have a more general methodology for analyzing optimal time-consistent policies.

In this paper, we show how to use optimal control theory to derive time-consistent government policies in nonlinear dynamic general equilibrium models. We do this by extending the insight of Cohen and Michel (1988), who showed that in *linear* models time-consistent policies can be found using optimal control theory by imposing a linear relationship between predetermined state variables and the costate variables from private agents' maximization problems. Here, we use an analogous procedure based on the Den Haan and Marcet (1990) technique of parameterized expectations, which replaces nonlinear functions of expected future costates by flexible functions of current states. This leads to a *nonlinear* relationship between current state and costate variables, which is verified in equilibrium to an arbitrarily close degree of approximation. The optimal control problem of the government is recursive (in a sense to be defined below), unlike the Ramsey problem which is common in the optimal taxation literature. The optimal policies found using this methodology have the characteristic that they depend only

[2]See Ambler and Paquet (1996, 1997), Ambler and Cardia (1997), Klein, Krusell and Rios Rull (2003); and Ambler (2000).

on current state variables, as do private agents' optimal feedback rules.[3]

The rest of the paper is structured as follows. In the following section, we develop an abstract model of the interaction between two agents (a representative private agent and a government). In the third section, we review how the time consistency problem arises by analyzing a Ramsey problem applied to the abstract model. In the fourth section, we present the Den Haan and Marcet methodology and show how to use it in conjunction with deriving an optmal time consistent feedback rule for the government. In the fifth section, we formally demonstrate the recursivity of the government's problem. In the sixth section, we summarize how to use the Den Haan and Marcet methodology to calculate a numerical solution to the optimal control problem. In the seventh section, we present a simple model of public investment in order to illustrate the technique. Conclusions are in the eighth section.

2 The Model

The economy consists of a representative household,[4] a representative competitive firm, and a government.[5] The household has an infinite planning horizon and maximizes its utility taking as given all relative prices and the

[3]We exclude more complex strategies that are history-dependent. Because optimal feedback rules are memoryless, the equiibrium in the models we consider is known as Markov-perfect equilibrium. See Bernheim and Ray (1989) and Maskin and Tirole (1993).

[4]The approach here could easily be extended to models of heterogeneous agents, but the notation would be too cumbersome for the purposes of this paper. See Rios Rull (1999) for a good introduction to heterogeneous agent models.

[5]Although the analysis is framed in terms of optimal government policy, it is clear that it could be used to derive time consistent feedback rules in any dynamic game with a Stackelberg leader.

government's policy rule. The government chooses its policies to maximize social welfare, which in this framework leads it to maximize the utility of the representative household, subject to the first order conditions of the household.

2.1 The Household

The utility function of the household can be writen as[6]

$$U = E_t \sum_{i=0}^{\infty} \beta^i r\left(z_{t+i}, g_{t+i}, S_{t+i}, s_{t+i}, D_{t+i}, d_{t+i}\right), \tag{1}$$

where z_t is a vector of exogenous state variables of dimension η_zx1, g_t is a η_gx1 vector of government policy variables, s_t is a η_sx1 vector of endogenous state variables under the control of the individual household, S_t is a η_sx1 vector of endogenous aggregate state variables, which are the aggregate counterparts of s_t, d_t is a η_dx1 vector of the household's control variables, D_t is a η_dx1 vector of the aggregate counterparts of d_t, and E_t denotes mathematical expectations conditional on information available at time t. The household chooses $\{d_{t+i}\}_{i=0}^{\infty}$ in order to maximize its utility, subject to the following set of constraints: the law of motion of the household's state variables,

$$s_{t+1} = b\left(z_t, g_t, S_t, s_t, D_t, d_t\right); \tag{2}$$

the law of motion of the aggregate state variables,

$$S_{t+1} = B\left(z_t, g_t, S_t, D_t, \right); \tag{3}$$

[6]The notation used is patterned after that of Hansen and Prescott (1995).

the feedback rule for the aggregate control variables,

$$D_t = D\left(z_t, g_t, S_t\right); \tag{4}$$

and the feedback rule for the government's policy variables,

$$g_t = g\left(z_t, S_t\right). \tag{5}$$

The assumption that the law of motion for the household's state variables is an explicit function for s_{t+1} is not innocuous. If there were an implicit relationship between s_{t+1} and current states and controls, the household's first order condition for the choice of d_t would depend on the future state of the economy and the government's problem. Solving for the private sector's control variables as an explicit function of current state variables and costate variables as in (10) below would no longer be possible. The solution to this problem leads to a feedback rule of the form

$$d_t = d\left(z_t, g_t, S_t, s_t\right). \tag{6}$$

As equilibrium conditions, we will impose *aggregate consistency conditions.* The laws of motion for S_t and s_t must satisfy

$$b\left(z_t, g_t, S_t, S_t, D_t, D_t\right) = B\left(z_t, g_t, S_t, D_t, \right), \tag{7}$$

and the feedback rules for D_t and d_t must be consistent:

$$d\left(z_t, g_t, S_t, S_t\right) = D\left(z_t, g_t, S_t\right). \tag{8}$$

The Lagrangian of the household's problem can be written as

$$\mathcal{L}_t = E_t \sum_{i=0}^{\infty} \beta^i \Big\{ r\left(z_{t+i}, g_{t+i}, S_{t+i}, s_{t+i}, D_{t+i}, d_{t+i}\right)$$

$$+\lambda_{t+i}\left[s_{t+i+1} - b\left(z_{t+i}, g_{t+i}, S_{t+i}, s_{t+i}, D_{t+i}, d_{t+i}\right)\right]\Bigg\}. \tag{9}$$

The household maximizes the Lagrangian by choosing $\{d_{t+i}, s_{t+i+1}, \lambda_{t+i}\}_{i=0}^{\infty}$. The first order conditions with respect to variables chosen at time t can be written as follows:

$$d_t: \quad \frac{\partial r(\cdot)}{\partial d_t} - \lambda_t \frac{\partial b(\cdot)}{\partial d_t} = 0,$$

$$s_{t+1}: \quad E_t\left(\lambda_t + \beta \frac{\partial r(\cdot)}{\partial s_{t+1}} - \beta \lambda_{t+1} \frac{\partial b(\cdot)}{\partial s_{t+1}}\right) = 0,$$

$$\lambda_t: \quad s_{t+1} = b\left(z_t, g_t, S_t, s_t, D_t, d_t\right).$$

When we impose the aggregate consistency constraints, the first order condition with respect to d_t gives a set of η_dx1 static equations. We assume that it is possible to solve the equations explicitly for D_t as a function of states and costates:

$$D_t = \tilde{D}\left(z_t, g_t, S_t, \lambda_t\right). \tag{10}$$

3 A Ramsey Problem

Models such as the one outlined in the previous section are often used to set up Ramsey (1927) problems, in which the government maximizes social welfare subject to constraints which guarantee that the first order conditions of households are satisfied. In the present context, this leads to the following

Lagrangian for the government's problem:[7]

$$\mathcal{L}_t^g = E_t \sum_{i=0}^{\infty} \beta^i \Big\{ r\left(z_{t+i}, g_{t+i}, S_{t+i}, S_{t+i}, D_{t+i}, D_{t+i}\right)$$

$$+\pi_{t+i}^1 \Big[S_{t+i+1} - b\left(z_{t+i}, g_{t+i}, S_{t+i}, S_{t+i}, D_{t+i}, D_{t+i}\right) \Big]$$

$$+\pi_{t+i}^2 \left[\frac{\partial r(\cdot)}{\partial d_{t+i}} - \lambda_{t+i} \frac{\partial b(\cdot)}{\partial d_{t+i}} \right]'$$

$$+\pi_{t+i}^3 \left[\lambda_{t+i} + \beta \frac{\partial r(\cdot)}{\partial s_{t+i+1}} - \beta \lambda_{t+i+1} \frac{\partial b(\cdot)}{\partial s_{t+i+1}} \right]' \Big\}, \tag{11}$$

where the household's control variables and state variables are replaced by their aggregate per capita counterparts. The government maximizes the representative agent's utility. This assumption is not necessary but it simplifies the analysis. The government maximizes the Lagrangian with respect to its control variables $\{g_{t+i}\}_{i=0}^{\infty}$, the aggregate equivalents of the private sector's control variables, and the Lagrange multipliers.

The force of the time inconsistency argument is made clear if we consider the first order conditions for the optimal choice of g_{t+1}. We have:

$$\frac{\partial \mathcal{L}_t^g}{\partial g_{t+1}} = 0 = E_t \Big\{ \beta \frac{\partial r(\cdot)}{\partial g_{t+1}} - \beta \pi_{t+1}^1 \frac{\partial B(\cdot)}{\partial g_{t+1}}$$

$$+\beta \pi_{t+1}^2 \left[\frac{\partial^2 r(\cdot)'}{\partial g_{t+1} \partial d_{t+1}} - \frac{\partial}{\partial g_{t+1}} \left(\frac{\partial b(\cdot)}{\partial d_{t+1}}' \lambda_{t+1}' \right) \right]$$

$$+\beta \pi_t^3 \left[\frac{\partial^2 r(\cdot)'}{\partial g_{t+1} \partial s_{t+1}} - \frac{\partial}{\partial g_{t+1}} \left(\frac{\partial b(\cdot)}{\partial s_{t+1}}' \lambda_{t+1}' \right) \right] \Big\} \tag{12}$$

[7]As noted in the introduction, it is often possible to simplify the government's Lagrangian using the primal approach. This approach is not applicable to the highly abstract model presented here.

The term in π_t^3 gives the influence of future policy on the *current* behavior of households, via its effect on the forward-looking costate variables λ_t. If we allow the government to reoptimize at time $t+1$, the first order conditions for the choice of g_{t+1} become:

$$\frac{\partial \mathcal{L}_{t+1}^g}{\partial g_{t+1}} = 0 = \beta \frac{\partial r^g(\cdot)}{\partial g_{t+1}} - \beta \pi_{t+1}^1 \frac{\partial B(\cdot)}{\partial g_{t+1}}$$
$$+\beta \pi_{t+1}^2 \Big[\frac{\partial^2 r(\cdot)'}{\partial g_{t+1} \partial d_{t+1}} - \frac{\partial}{\partial g_{t+1}} \left(\frac{\partial b(\cdot)}{\partial d_{t+1}}' \lambda_{t+1}' \right) \Big] \quad (13)$$

Bygones are bygones. The effect of the government's controls at time $t+1$ on the household's actions at time t no longer appears. Even in the absence of unanticipated shocks, the government will in general revise its optimal plans.

Since the values of the private sector's costate variables λ_t are not pinned down by initial conditions, one of optimality conditions for the government's problem has to be

$$\pi_t^3 = 0.$$

The private sector's costates give the marginal value of the state variables to the representative agent's utility. Since the government's welfare function is just the utility function of the representative agent, a necessary conditon to maximize welfare is that the contribution of a marginal change in these costates to welfare be zero. The future values of π_{t+i}^3, $i > 0$ are determined by the endogenous dynamics of the economy. After time t, they will only be zero by coincidence. However, if the government is allowed to reoptimize at time $t+i$, with $i > 0$, it will once again want to set

$$\pi_{t+i}^3 = 0.$$

In so doing, its optimal strategy changes. Time inconsistency arises here because the government's problem is not *recursive*, in the sense of Sargent (1987, p.19). An agent's problem is recursive if its control variables dated t influence states dated $t+1$ and later and influence returns dated t and later. The household's current actions depend partly on its expectations of future government actions. In the Ramsey problem, the government's announced or future policies influence private agents' current behavior and therefore the current return via the function $r(\cdot)$.

4 The Den Haan and Marcet Methodology

Den Haan and Marcet (1990) propose finding a solution to nonlinear dynamic general equilibrium models replacing expectations of nonlinear functions of future state variables by a flexible functional form that depends on current exogenous and endogenous state variables and parameters.

In the present context, this involves replacing the last two terms associated with the π_t^3 constraint in the government's problem as follows:

$$\text{Let} \quad \beta E_t \left(\frac{\partial r(\cdot)}{\partial s_{t+1}} - \lambda_{t+1} \frac{\partial b(\cdot)}{\partial s_{t+1}} \right)' = \phi\left(z_t, S_t\right). \tag{14}$$

A byproduct of this is that λ_t, the vector of costate variables, becomes just a function of the current exogenous and endogenous state variables:

$$\lambda_t = -\phi\left(z_t, S_t\right)'. \tag{15}$$

This is exactly the form of the constraint imposed (in the context of *linear* models) by Cohen and Michel (1988). Making the government's problem subject to this additional constraint ties its hands. It is not allowed to choose its policy in order to set the initial values of the costates equal to zero. If allowed to reoptimize, it is not tempted to change its policy in order to bring the costates back to zero.

5 The Recursivity of the Government's Problem

We now assume that the government maximizes the utility of the representative private agent, as in the Ramsey problem described above, subject to the additional constraint given by the parameterized expectations in (15). Using parameterized expectations, we can show that the government's problem becomes recursive. We can write it as a dynamic programming problem in which the period-t return function does not depend on the future values of the government's controls.

We need one further assumption to demonstrate recursivity. We assume that the set of η_g equations associated with the π_t^2 constraint, once λ_t is replaced by $-\phi\left(z_t, S_t\right)$, can be solved out to find an explicit set of feedback rules for D_t which in equilibrium is just equation (4). Then, we have

Proposition: Subject to the constraint imposed by parameterized expectations, the government's maximization problem is recursive.

Proof: Substituting in parameterized expectations, we have the following expression for the difference between the government's Lagrangian at time t and the discounted value of its Lagrangian at time $t+1$, which gives the government's one-period return function:

$$\mathcal{L}_t^g - \beta E_t \mathcal{L}_{t+1}^g = r\left(z_t, g_t, S_t, S_t, D_t, D_t\right)$$

$$+\pi_t^1 \left(S_{t+1} - b\left(z_t, g_t, S_t, S_t, D_t, D_t\right)\right)$$

$$+\pi_t^2 \left(\frac{\partial r(\cdot)}{\partial d_t} + \phi(z_t, S_t)' \frac{\partial b(\cdot)}{\partial d_t}\right). \qquad (16)$$

The one-period return function of the government does not depend directly or indirectly on g_{t+1}, since we suppose that D_t can be written as a function of only current state variables and g_t. The government's value function can be written as

$$V_t^g\left(z_t, S_t\right) = \max_{g_t} \Big\{ r\left(z_t, g_t, S_t, S_t, D\left(z_t, g_t, S_t\right), D\left(z_t, g_t, S_t\right)\right)$$

$$+\beta E_t\left(V_{t+1}^g\left(z_{t+1}, S_{t+1}\right)\right) \Big\}, \qquad (17)$$

q.e.d.

The maximization is subject to the law of motion of the aggregate state variables S_t, and to the first order conditions for the household's choice of its controls d_t, with the household's Lagrange multipliers λ_t substituted out using the constraint given in (15). Note that the government's problem becomes recursive partly because one of the underlying assumptions of this approach is that there is a time-invariant feedback rule for D_t which depends only on the current state of the economy. This leads to a feedback rule for the government compatible with (5) that depends only on the current state of the economy. It is as if the current government derives its optimal policy using dynamic programming techniques, under the assumption that all future governments will derive their optimal policies in the same way.[8]

In the context of the Ramsey problem described earlier, we have instead

$$\mathcal{L}_t^g - \beta E_t \mathcal{L}_{t+1}^g = r\left(z_t, g_t, S_t, S_t, D_t, D_t\right)$$

$$+\pi_t^1 \left(S_{t+1} - b\left(z_t, g_t, S_t, S_t, D_t, D_t\right)\right)$$
$$+\pi_t^2 \left(\frac{\partial r(\cdot)}{\partial d_t} - \lambda_t \frac{\partial b(\cdot)}{\partial d_t}\right)'$$
$$+\pi_t^3 \left(\lambda_t + \beta \frac{\partial r(\cdot)}{\partial s_{t+1}} - \beta \lambda_{t+1} \frac{\partial b(\cdot)}{\partial s_{t+1}}\right)'.$$

Because of the presence of the future value of the household's constraint λ_{t+1}, the government's problem fails to be recursive.

6 Numerical Solution

In the present context, using dynamic programming techniques to solve the government's maximization problem is not convenient. The form of the government's value function is not known, and would have to be approximated by using one of many well-known techniques (quadratic approximation of the value function around the steady state equilibrium, discretization of the state space, etc.).[9] The use of parameterized expectations in conjunction

[8]One interpretation of optimal time consistent policy is that the current government is playing a game against the private sector *and* future governments, taking the feedback rules of the future governments as given.

[9]Krusell and Smith (2000) develop a different method of approximating the value function using envelope conditions and evaluating higher-order derivatives of the policy functions of the government and the private sector.

with control theory allows for an arbitrarily close approximation to the exact solution to the underlying problem.

Using the Lagrangian in (11) above, after substituting out λ_t using parameterized expectations and eliminating the third constraint, the first order conditions for the government's problem can be written as follows:

$$\frac{\partial \mathcal{L}_t^g}{\partial g_t} = 0 = \beta \frac{\partial r^g(\cdot)}{\partial g_t} - \pi_t^1 \frac{\partial B(\cdot)}{\partial g_t}$$
$$+\pi_t^2 \Big[\frac{\partial^2 r(\cdot)'}{\partial g_t \partial d_t} + \frac{\partial}{\partial g_t} \left(\frac{\partial b(\cdot)}{\partial d_t}' \phi(z_t, S_t) \right) \Big], \tag{18}$$

$$\frac{\partial \mathcal{L}_t^g}{\partial S_{t+1}} = 0 = E_t \Bigg\{ \pi_t^1 + \beta \frac{\partial r(\cdot)}{\partial S_{t+1}} - \beta \pi_{t+1}^1 \frac{\partial B(\cdot)}{\partial S_{t+1}}$$
$$+\beta \pi_{t+1}^2 \frac{\partial^2 r(\cdot)}{\partial S_{t+1} \partial d_{t+1}} - \beta \pi_{t+1}^2 \frac{\partial}{\partial g_{t+1}} \left(\frac{\partial b(\cdot)}{\partial d_{t+1}}' \phi(z_t, S_t) \right) \Bigg\} \tag{19}$$

$$\frac{\partial \mathcal{L}_t^g}{\partial D_t} = 0 = \beta \frac{\partial r^g(\cdot)}{\partial D_t} - \pi_t^1 \frac{\partial B(\cdot)}{\partial D_t}$$
$$+\pi_t^2 \Big[\frac{\partial^2 r(\cdot)'}{\partial D_t \partial d_t} + \frac{\partial}{\partial D_t} \left(\frac{\partial b(\cdot)}{\partial d_t}' \phi(z_t, S_t) \right) \Big]. \tag{20}$$

Note that this leads to a time-autonomous set of nonlinear difference equations. In principle, the system is saddle-point stable. The initial conditions of the government's costate variables are those that place the system on the multi-dimensional convergent manifold of the system. The initial conditions of the costates therefore depend on the current state of the economy, given by the values of z_t and S_t.[10] We can therefore suppose that

$$E_t \Bigg\{ \beta \frac{\partial r(\cdot)}{\partial S_{t+1}} - \beta \pi_{t+1}^1 \frac{\partial B(\cdot)}{\partial S_{t+1}}$$

[10] In the Ramsey problem, the optimality condition that the private agent's costates be equal to zero at the moment the government optimizes, independently of the state of the economy, means that the resulting dynamical equation system is not time-autonomous.

$$
+\beta\pi_{t+1}^{2}\frac{\partial^{2}r(\cdot)}{\partial S_{t+1}\partial d_{t+1}}-\beta\pi_{t+1}^{2}\frac{\partial}{\partial g_{t+1}}\left(\frac{\partial b(\cdot)}{\partial d_{t+1}}^{'}\phi\left(z_{t},S_{t}\right)\right)\Bigg\}^{'}
$$

$$
=\psi\left(z_{t},S_{t}\right). \tag{21}
$$

We have

$$
\pi_{t}^{1}=-\psi\left(z_{t},S_{t}\right)^{'}. \tag{22}
$$

In a variation of the methodology described by Den Haan and Marcet (1990) and Marcet and Lorenzoni (1999), the model can be simulated using the following steps:

- Parameterize the $\phi(\cdot)$ and $\psi(\cdot)$ functions using flexible functional forms such as polynomials or orthogonalized polynomials.

- Initialize the parameter values of the expectations functions.

- For given parameter values of the parameterized expectations functions, simulate the model for a large number of time periods. Aside from the laws of motion for S_t and z_t, all of the equations that need to be solved are static. The laws of motion themselves are recursive.

- Estimate the parameters in the $\phi(\cdot)$ and $\psi(\cdot)$ functions by nonlinear regression, with the dependent variables being the series generated by numerical simulation, in order to minimize the sum of squared expectational errors.

Its optimal policy is not a time-invariant function of the state of the economy, but rather depends on the time since it last optimized. This is another way of interpreting the time inconsistency of optimal policy in the Ramsey problem.

- Repeat the simulation and estimation steps until the change in the parameters of the expectations functions between iterations is sufficiently small.

7 Application

We apply the techniques developed in above to a simple model of optimal public spending. The utility function of the representative private agent depends on both his own private consumption spending and on government purchases. The government chooses public spending in order to maximize social welfare, which is just the expected utility of the representative private agent, financing this spending via a proportional tax on total income.

The representative private agent maximizes expected utility, which is given by

$$U = E_t \sum_{i=o}^{\infty} \beta^i \left\{ \ln(c_{t+i}) + \mu \ln(g_{t+i}) \right\}, \tag{23}$$

where c_t is private consumption and g_t is public spending. The private agent holds the capital stock and rents it to firms. Its period budget constraint is given by

$$(1 - \tau_t)\left(w_t + (r_t - \delta)k_t\right) + k_t = c_t + k_{t+1}, \tag{24}$$

where w_t is the competitive real wage, r_t is the competitive real rental rate of capital, k_t is capital held by the individual, and τ_t is the rate of taxation on total income. The time endowment of the individual is normalized to equal one, so that before-tax labor income is just given by w_t.

The aggregate production function is given by

$$y_t = a_t {k_t}^{\alpha}, \tag{25}$$

where y_t is GDP. The law of motion for a_t is given by

$$a_t = \rho a_{t-1} + \varepsilon_t, \tag{26}$$

where ε_t is a white noise shock with variance σ_{ε}^2.

The government finances public investment via a linear tax on total income. We rule out lump sum taxation in order to make the policy problem one of finding the second-best outcome, which leads to a distinction between time-consistent policies and time inconsistent policies. The government's budget is balanced in each period, so that

$$\tau_t \left(w_t + (r_t - \delta) k_t \right) = g_t. \tag{27}$$

The individual's first order conditions for utility maximization imply:

$$\frac{1}{c_t} = \lambda_t, \tag{28}$$

$$\lambda_t = \beta E_t \left(\lambda_{t+1} \left[1 + (1 - \tau_{t+1})(r_{t+1} - \delta) \right] \right), \tag{29}$$

$$k_{t+1} = (1 - \tau_t) y_t + (1 - \delta) k_t - c_t \tag{30}$$

The government's maximization problem can be expressed as follows:

$$\mathcal{L} = E_t \sum_{i=0}^{\infty} \beta^i \Bigg\{ \ln(c_{t+i}) + \mu \ln(\tau_{t+i}) + \mu \ln(y_{t+i} - \delta k_{t+i})$$

$$+ \pi_{t+i}^1 \left((1 - \delta) k_{t+i} + y_{t+i} - \tau_{t+i}(y_{t+i} - \delta k_{t+i}) - c_{t+i} - k_{t+i+1} \right)$$

$$+\pi^2_{t+i}\left(\frac{1}{c_{t+i}}-\frac{\beta}{c_{t+i+1}}\left((1-\tau_{t+i+1})\,\alpha\frac{y_{t+i+1}}{k_{t+i+1}}+1-\delta(1-\tau_{t+i+1})\right)\right)\Bigg\} \quad (31)$$

The first-order conditions imply:

$$\tau_t:\quad \frac{\mu}{c_t}-\pi^1_t(y_t-\delta k_t)+\frac{\pi^2_{t-1}}{c_t}\left(\alpha\frac{y_t}{k_t}-\delta\right)=0,$$

$$c_t:\quad \frac{1}{c_t}-\pi^1_t-\frac{\pi^2_t}{(c_t)^2}+\pi^2_{t-1}\left((1-\tau_t)\,\alpha\frac{y_t}{k_t}+1-\delta(1-\tau_t)\right)=0,$$

$$\pi^1_t:\quad k_{t+1}=(1-\tau_t)y_t+(1-\delta(1-\tau_t))k_t-c_t,$$

$$\pi^2_t:\quad \frac{1}{c_t}-\beta E_t\left(\frac{1}{c_{t+1}}\left((1-\tau_{t+1})\,\alpha\frac{y_{t+1}}{k_{t+1}}+1-\delta(1-\tau_{t+1})\right)\right)=0,$$

$$\begin{aligned} k_{t+1}\quad:\quad & \pi^1_t=\beta E_t\left(\frac{\alpha(\alpha-1)}{c_{t+1}}\frac{(1-\tau_{t+1})y_{t+1}}{k^2_{t+1}}\pi^2_t\right) \\ & +\beta E_t\left(\pi^1_{t+1}\left(1+(1-\tau_{t+1})\left(\alpha\frac{y_{t+1}}{k_{t+1}}-\delta\right)\right)+\mu\frac{\alpha\frac{y_{t+1}}{k_{t+1}}-\delta}{y_{t+1}-\delta k_{t+1}}\right),\end{aligned}$$

where y_t and a_t are defined respectively in (25) and (26).

As explained in previous sections, the Ramsey solution is time-inconsistent. The time-consistent solution can be found by imposing a nonlinear constraint between the predetermined state variables and the co-state variables. Specifically, the Euler equation is replaced by

$$\frac{1}{c_t}-\phi(a_t,k_t)=0,$$

where ϕ is a function to approximate using the PEA method.

The Lagrangian can be written as:

$$\begin{aligned}\mathcal{L} \;=\; & E_t\sum_{i=0}^{\infty}\beta^i\Bigg\{\ln(c_{t+i})+\mu\ln(\tau_{t+i})+\mu\ln(y_{t+i}-\delta k_{t+i}) \\ & +\pi^1_{t+i}\left((1-\delta)k_{t+i}+y_{t+i}-\tau_{t+i}(y_{t+i}-\delta k_{t+i})-c_{t+i}-k_{t+i+1}\right) \\ & +\pi^2_{t+i}\left(\frac{1}{c_{t+i}}-\phi(a_t,k_t)\right)\Bigg\}.\end{aligned}$$

The first-order conditions imply:

$$\tau_t: \quad \frac{\mu}{c_t} - \pi_t^1(y_t - \delta k_t) = 0,$$

$$c_t: \quad \frac{1}{c_t} - \pi_t^1 - \frac{\pi_t^2}{(c_t)^2} = 0,$$

$$\pi_t^1: \quad k_{t+1} = (1-\tau_t)y_t + (1-\delta(1-\tau_t))k_t - c_t,$$

$$\pi_t^2: \quad \frac{1}{c_t} - \phi(a_t, k_t) = 0,$$

$$k_{t+1} \quad : \quad \pi_t^1 = \beta E_t \Bigg\{ \pi_{t+1}^1 \left(1 + (1-\tau_{t+1})\left(\alpha \frac{y_{t+1}}{k_{t+1}} - \delta\right)\right) - \pi_{t+1}^2 \frac{\partial \phi}{\partial k_{t+1}}(a_{t+1}, k_{t+1}) + \mu \frac{\alpha \frac{y_{t+1}}{k_{t+1}} - \delta}{y_{t+1} - \delta k_{t+1}} \Bigg\}.$$

In what follows, we assume the rate of depreciation is equal to zero ($\delta = 0$). The parameter values used to simulate the model are summarized in Table 1. For the most part, they are standard values used in the real business cycle literature.

7.1 Simulation Results

To solve the time-consistent and Ramsey problems, we follow the methodology of Den Haan and Marcet (1990), and Marcet and Lorenzoni (1999). In both cases, we need to find two interpolating functions (for each Euler equation). Let us describe the methodology for the time-consistent problem. There are two state variables, k_t and a_t, so that the two interpolating functions, ϕ and ψ, should be functions of both k_t and a_t, and verify

$$\frac{1}{c_t} - \phi(a_t, k_t) = 0 \quad \text{and} \quad \pi_t^1 - \psi(a_t, k_t) = 0.$$

Table 1: Model Parameter Values

Parameter	Value
β	0.99
α	0.33
μ	0.5
δ	0.00
ρ	0.95
σ_ε	0.01

We make the guess that

$$\phi(a_t, k_t; \theta) = \exp(\theta' P(a_t, k_t))$$

$$= \exp(\theta_0 + \theta_1 \log k_t + \theta_2 \log a_t + \theta_3 (\log k_t)^2 + \theta_4 (\log a_t)^2 + \theta_5 (\log k_t)(\log a_t)))$$

and

$$\psi(a_t, k_t; \gamma) =$$

$$\exp(\gamma_0 + \gamma_1 \log k_t + \gamma_2 \log a_t + \gamma_3 (\log k_t)^2 +$$

$$\gamma_4 (\log a_t)^2 + \gamma_5 (\log k_t)(\log a_t)).$$

One potential problem with such a functional form is precisely related to the fact that it uses simple polynomials which then may generate multicolinearity problem during the estimation set. In this respect, since consumption is a time-varying fraction of the output (defined here a Cobb-Douglas production function), we check the robustness of our results by assuming that

$\phi(a_t, k_t; \theta) = \theta_0 k_t^{\theta_1} a_t^{\theta_2}$ and $\psi(a_t, k_t; \theta) = \gamma_0 k_t^{\gamma_1} a_t^{\gamma_2}$ (see Den Haan and Marcet, 1990).

A second problem that arises in this approach is to select initial conditions for each parameter vector. In effect, this step is crucial for, at least three reasons. First, the problem is fundamentally nonlinear and thus different initial conditions may lead to alternative dynamics. Second, convergence is not always guaranteed. Third, economic theory imposes a set of restrictions to insure positivity of some variables, for example $c_t \geq 0$ and $1 \geq \tau_t \geq 0$.

A third problem is related to the choice of the smoothing paramater (see Marcet and Lorenzoni, 1999, for a discussion). In effect, at the ith step, the new parameter vector is defined by $\theta^{(i)} = \lambda\widehat{\theta} + (1-\lambda)\theta^{(i-1)}$, where $\theta^{(i-1)}$ and $\widehat{\theta}$ are, respectively, the parameter vector at the (i-1)th step, and the estimated paramaters resulting of the regression between the P.E.A-simulated data and the model-simulated data. The speed of convergence mainly depends on λ.

The stopping criterion was set at $\left|\theta_j^{(i)} - \theta_j^{(i-1)}\right| \leq 10^{-6}$ and $\left|\gamma_j^{(i)} - \gamma_j^{(i-1)}\right| \leq 10^{-6}$, $\forall j$, and 20,000 data points were used to compute the nonlinear regressions.

Initial conditions were set as follows. We first solve the time-inconsistency problem relying on a log-linear approximation. We then generate a random draw of size T for the error terms and generates series using the log-linear approximation solution. We solve the P.E.A. problem for the Ramsey problem and then use as initial conditions in the time-consistent problem, the corresponding coefficients of the time-inconsistency decision rules. Therefore, we make the assumption that the time-inconsistency solution provides a good approximation for our problem. As long as the final decision rules do differ form the initial conditions, this choice may not affect our results.

[Description of results: TO BE COMPLETED]

8 Conclusions

The methodology proposed in this paper is quite general, and leads to systems of dynamical equations which can easily be simulated with available computer technology and relatively parsimonious representations of the parameterized expectations functions (with 2xν_p free parameters to estimate, where ν_p is the order of the polynomials used in the expectations functions). Deriving time-consistent government policies using these methods is conceptually as straightforward as solving Ramsey problems. The technique should allow researchers to do normative analysis, comparing the levels of welfare attainable with and without precommitment by the government. It should also be useful for positive analysis, for example comparing the predictions of a given model for comovements between government policy variables and other macroeconomic aggregates with and without precommitment. As suggested by Judd (1998), with current advances in computer technology it should become more and more common to use numerical methods to advance our understanding of economic theory.

References

[INCOMPLETE – references to be reconciled with rest of paper.]

Aiyagari, Rao and R. Anton Braun (1997), "Some Models to Guide Monetary Policy Makers", mimeo, Federal Reserve Bank of Minneapolis

Aiyagari, Rao and Ellen McGrattan (1995), "The Optimum Quantity of Debt", Staff Report 203, Federal Reserve Bank of Minneapolis

Ambler, Steve (2000), "Optimal Time-Consistent Fiscal Policy with Overlapping Generations", mimeo

Ambler, Steve and Emanuela Cardia (1997), "Optimal Government Spending in a Business Cycle Model", *Business Cycles and Macroeconomic Stability: Should We Rebuild Built-In Stabilizers?* Jean-Olivier Hairault, Pierre-Yves Hénin and Franck Portier, editors, Kluwer Academic Press, 31-53

Ambler, Steve and Alain Paquet (1996), "Fiscal Spending Shocks, Endogenous Government Spending and Real Business Cycles", *Journal of Economic Dynamics and Control* 20, 237-256

Ambler, Steve and Alain Paquet (1997), "Recursive Methods for Computing Equilibria of General Equilibrium Dynamic Stackelberg Games", *Economic Modelling* 14, 155-173

Atkinson, Anthony B. and Joseph E. Stiglitz (1980), *Lectures on Public Economics.* New York, McGraw-Hill

Auerbach, Alan and Laurence Kotlikoff (1987), *Dynamic Fiscal Policy.* Cambridge, Cambridge University Press

Barro, Robert J. (1979), "On the Determination of Public Debt", *Journal of Political Economy* 87, 940-971

Benhabib, Jess and Aldo Rustichini (1997), "Optimal Taxes without Commitment", *Journal of Economic Theory* 77, 231-259

Benhabib, Jess, Aldo Rustichini and Andrés Velasco (1996), "Public Capital and Optimal Taxes without Commitment", C.V. Starr Center working paper 96-19, New York University

Benhabib, Jess and Andrés Velasco (1996), "On the Optimal and Best Sustainable Taxes in an Open Economy", *European Economic Review* 40, 135-154

Bernheim, B. Douglas and Debraj Ray (1989), "Markov Perfect Equilibria in Altruistic Growth Economies with Production Uncertainty", *Journal of Economic Theory* 47, 195-202

Benigno Pierpaolo and Michael Woodford (2004), "Optimal Stabilization Policy When Wages and Prices are Sticky: The Case of a Distorted Steady State" NBER working paper 10839

Benigno Pierpaolo and Michael Woodford (2005), "Optimal Taxation in an RBC Model: A Linear-Quadratic Approach" NBER working paper 11029

Braun, R. Anton (1994), "Tax Disturbances and Real Economic Activity in the Postwar United States", *Journal of Monetary Economics* 33, 441-462

Calvo, Guillermo and Maurice Obstfeld (1988), "Optimal Time-Consistent Fiscal Policy with Finite Lifetimes: Analysis and Extensions" in Elhanan Helpman, Assaf Razin and Efraim Sadka, eds., *Economic Effects of the Government Budget*. Cambridge, MA, MIT Press

Cassou, S.P. and Kevin J. Lansing (1998), "Optimal Fiscal Policy, Public Capital, and the Productivity Slowdown", *Journal of Economic Dynamics and Control* 22, 911-935

Chamley, C.P. (1985), "Efficient Taxation in a Stylized Model of Intertemporal General Equilibrium", *International Economic Review* 26, 451-468

Chamley, C.P. (1986), "Optimal Taxation of Income in General Equilibrium with Infinite Lives", *Econometrica* 54, 607-622

Chari, V.V., Lawrence J. Christiano and Patrick J. Kehoe (1991), "Optimal Fiscal and Monetary Policy: Some Recent Results", *Journal of Money, Credit and Banking* 23, 519-539

Chari, V.V., Lawrence J. Christiano and Patrick J. Kehoe (1994), "Optimal Fiscal Policy in a Business Cycle Model", *Journal of Political Economy* 102, 617-652

Chari, V.V., Lawrence J. Christiano and Patrick J. Kehoe (1995), "Policy Analysis in Business Cycle Models" in Thomas F. Cooley, ed., *Frontiers of Business Cycle Research*. Princeton, Princeton University Press

Chari, V.V. and Patrick J. Kehoe (1990), "Sustainable Plans", *Journal of Political Economy* 98, 783-802

Chari, V.V. and Patrick J. Kehoe (1998), "Optimal Fiscal and Monetary Policy", Staff Report 251, Federal Reserve Bank of Minneapolis, July

Cohen, Daniel and Philippe Michel (1988), "How Should Control Theory Be Used to Calculate a Time-Consistent Government Policy?", *Review of Economic Studies* 55, 263-274

Currie, David and Paul Levine (1993), *Rules, Reputation and Macroeconomic Policy Coordination.* Cambridge, Cambridge University Press

Den Haan, W.J. and A. Marcet (1990), "Solving the Stochastic Growth Model by Parameterizing Expectations", *Journal of Business and Economic Statistics* 8, 31-34

Domeij, David and Paul Klein (1998), "Pre-Announced Optimal Tax Reform", mimeo, Institute for International Economic Studies, University of Stockholm

Erosa, Andrés and Martin Gervais (1998), "Optimal Taxation in Life-Cycle Economies", mimeo, University of Western Ontario

Giannoni, Marc and Michael Woodford (2002), "Optimal Interest-Rate Rules: I. General Theory", NBER working paper 9419

Hall, Robert E. and Alvin Rabushka (1995), *The Flat Tax.* 2nd edition, Stanford, Hoover Institution Press

Hansen, Gary D. and Edward C. Prescott (1995), "Recursive Methods for Computing Equilibria of Business Cycle Models" in Thomas F. Cooley (ed.), *Frontiers of Business Cycle Research.* Princeton, Princeton University Press

James, Steven (1994), "Debt Reduction with Distorting Taxes and Incomplete Ricardianism: A Computable Dynamic General Equilibrium Analysis", in William Robson and William Scarth (eds.), *Deficit Reduction: What Pain, What Gain?.* Toronto, C.D. Howe Institute

Jones, L.E., R. Manuelli and P.E. Rossi (1993), "Optimal Taxation in Models of Endogenous Growth", *Journal of Political Economy* 101, 485-517

Jones, L.E., R. Manuelli and P.E. Rossi (1997), "On the Optimal Taxation of Capital Income", *Journal of Economic Theory* 73, 93-117

Judd, Kenneth L. (1985), "Redistributive Taxation in a Simple Perfect Foresight Model", *Journal of Public Economics* 28, 59-83

Judd, Kenneth L. (1987), "The Welfare Cost of Factor Taxation in a Perfect Foresight Model", *Journal of Political Economy* 95, 675-709

Judd, Kenneth L. (1998), *Numerical Methods in Economics.* Cambridge, MA, MIT Press

Judd, Kenneth L. (1999), "Optimal Taxation and Spending in General Competitive Growth Models", *Journal of Public Economics* 71, 1-26

Klein, Paul, Per Krusell and José-Victor Rios-Rull (2003), "Time-Consistent Optimal Fiscal Policy", mimeo, Institute for International Economic Studies

Krusell, Per, Vincenzo Quadrini and Jose-Victor Rios-Rull (1996), "Are Consumption Taxes Really Better than Income Taxes?", *Journal of Monetary Economics* 37

Krusell, Per and Anthony Smith (2000), "Equilibrium Welfare and Government Policy with Quasi-Ceometric Discounting", mimeo, University of Rochester.

Kydland, Finn E. and Edward C. Prescott (1980), "Dynamic Optimal Taxation, Rational Expectations, and Optimal Control", *Journal of Economic Dynamics and Control* 2, 79-91

Lansing, Kevin (1998), "Optimal Fiscal Policy in a Business Cycle Model with Public Capital", *Canadian Journal of Economics* 31, 337-64

Marcet, A. and G. Lorenzoni (1999), "The Parameterized Expectations Approach: Some Practical Issues", in R. Marimon and A. Scott, eds., *Computational Methods for the Study of Dynamic Economies* Oxford, Oxford University Press

Maskin, E. and J. Tirole (1993), "Markov Perfect Equilibrium", mimeo, Harvard University

Miller, Marcus and Mark Salmon (1985), "Policy Coordination and the Time Inconsistency of Optimal Policies in Open Economies", *Economic Journal* 95, 124-137

Ramsey, Frank (1927), "A Contribution to the Optimal Theory of Taxation", *Economic Journal* 37, 47-61

Ramsey, Frank (1928), "A Mathematical Theory of Saving", *Economic Journal* 38, 543-559

Rios-Rull, José-Victor (1995), "Models with Heterogeneous Agents", in T.F. Cooley, ed., *Frontiers of Business Cycle Research.* Princeton, Princeton

University Press

Rios-Rull, José-Victor (1999), “XX”, in R. Marimon and A. Scott, eds., *Computational Methods for the Study of Dynamic Economies*. Oxford, Oxford University Press

Sargent, Thomas J. (1987), *Dynamic Macroeconomic Theory*. Cambridge, MA, Harvard University Press

Scarth, William (1999), “Alternatives for Raising Living Standards”, mimeo, McMaster University

Slemrod, Joel (1990), “Optimal Taxation and Optimal Tax Systems”, *Journal of Economic Perspectives* 4, 157-178

Söderlind, Paul (1999), “Solution and Estimation of RE Macromodels with Optimal Policy”, *European Economic Review* 43, 813-823

Ventura, Gustavo (1996), “Flat Tax Reform: A Quantitative Assessment”, mimeo, University of Illinois

Woodford, Michael (2003), *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton, Princeton University Press

Zhu, X. (1992), “Optimal Fiscal Policy in a Stochastic Growth Model”, *Journal of Economic Theory* 58, 250-289