The Fractional Ornstein-Uhlenbeck Process: Term Structure Theory and Application*

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Abstract

The paper revisits dynamic term structure models (DTSMs) and proposes a new way in dealing with the limitation of the classical affine models. In particular, this paper expands the flexibility of the DTSMs by applying a fractional Brownian motion as the governing force of the state variable instead of the standard Brownian motion. This is a new direction in pricing non defaultable bonds with offspring in the arbitrage free pricing of weather derivatives based on fractional Brownian motions. By applying fractional Itô calculus and a fractional version of the Girsanov transform, a no arbitrage price of the bond is recovered by solving a fractional version of the fundamental bond pricing equation. Besides this theoretical contribution, the paper proposes an estimation methodology based on the Kalman filter approach, which is applied to the US term structure of interest rates.

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1 Introduction

Since the works of Vasicek (1977) and Cox, Ingersoll & Ross (1985) the literature devoted to estimating dynamic term structure models (DTSMs) has increased considerably. To capture the rich dynamics of observed yields, researchers are continually developing more complex models with greater flexibility.

Much of the literature of DTSMs focuses on the affine models as defined in e.g. Duffie & Kan (1996). Here interest rates are governed by multiple factors (state variables), and the affine models have the computational convenience that bond yields are linear functions of these variables. Two simplifying assumptions have been commonly used. First, the market price of risk was assumed to be a multiple of the instantaneous interest rate volatility. Second, the state variables were assumed to be independent, see Chen & Scott (1993), Pearson & Sun (1994), Chen & Scott (1995), Duan & Simonato (1999), Duffee (1999), and Geyer & Pichler (1999). However, Duffee (2002) showed that the restriction on the market price of risk results in bonds' excess returns exhibit unrealistic behaviour. Furthermore, Dai & Singleton (2000) refute the assumption of independent state variables based on empirical findings, and Aït-Sahalia (1996), Pfann, Schotman & Tschernig (1996), Conley, Hansen, Luttmer & Scheinkman (1997), and Stanton (1997) find evidence of nonlinearities in the drift of the interest rate process, which is inconsistent with the affine models.

In response to this, recent research has considered models with correlated state variables (Dai & Singleton (2000)), nonlinear dynamics or more flexible forms of the market price of risk (see, for example, Ahn & Gao (1999), Duarte (2004), Leippold & Wu (2000), and Ahn, Dittmer & Gallant (2002)), quadratic models (see Leippold & Wu (2000) and Ahn et al. (2002)), bond prices with jumps (Johannes (2004), Zhou (2001), and Das (2002)), and regime shifts (Ang & Bekaert (2002)).

This paper takes a different path. We expand the flexibility of the model by applying a fractional Brownian motion (fBm) as the governing force of the state variable instead of the usual Brownian motion, but still embed our model in the settings of the class of affine DTSMs. In particular, we use the fractional Ornstein-Uhlenbeck process, which is the fractional version of the classical Vasicek model, since the volatility function is driven by an fBm. This is a new direction in pricing non defaultable bonds with offspring in the arbitrage free pricing of weather derivatives based on fBm, see Brody, Syroka & Zervos (2002) and Benth (2003). Using fractional Itô calculus, (see Duncan, Hu & Pasik-Duncan (2000), Hu & Øksendal (2003), and Benth (2003)) and a fractional version of the Girsanov transform, we derive a no arbitrage price of the bond by solving a fractional version of the fundamental bond pricing equation. Besides this theoretical contribution, we propose an estimation methodology based on the Kalman filter approach and apply it to weekly observations on US Treasury rates from 1991 - 2002. Compared to using the classical Vasicek model, we find that the fractional version implies that the instantaneous interest rate is fractionally integrated of order d ($I(d)$) with $d = 0.05$ and mean reversion $\kappa = 0.22$. However, since κ is relatively small, the predicted yields are difficult to distinguish from $I(1.05)$ processes using unit root type tests.[1]

This combination of d and κ allows a fairly precise prediction of all the bond yields used in the analysis since the deviations to the observed yields are small for all maturities. In particular, the average absolute bias for the fractional model is 4.72 basis points (bp) while it is 12.01 (bp) for the ordinary Vasicek model. This improvement is obtained with only a minor increase in the average RMSE.

The remainder of the paper is organized as follows. In the next section we present the concept of fBm and fractional Itô calculus. In section 3, we expand the classical framework of arbitrage free pricing to cover the case where the state variable is governed by an fBm, and section 4 presents the fractional Ornstein-Uhlenbeck process (FOUP) as the natural extension of the Vasicek model. Section 5 recalls the one factor interest rate model and the classical bond pricing equation, and in section 6 we expand the framework to fractionally integrated processes and derive a fractional version of the arbitrage free bond price equation when the state variable is an FOUP. In section 7, we propose an estimation methodology based on the Kalman filter and describe its implementation. This setup is applied to weekly observations on US Treasury rates in section 8, and finally, section 9 offers some concluding remarks.

2 The Fractional Brownian Motion

The fractional Brownian Motion (fBm) is one of the simplest stochastic processes that exhibits long range dependence (or long memory). Since its introduction by Kolmogorov (1940, 1941), and its study by Mandelbrot & van Ness (1968), the fBm has been widely used in various areas of applications such as turbulence, finance, and telecommunication, see Mandelbrot & van Ness (1968), Brody et al. (2002), Hu, Øksendal & Sulem (2000), Benth (2003). The successes of the applications are mainly due to the self similar nature of the Gaussian fBm and its stationary increments, the fractional Gaussian noise.

The fBm is characterized by its Hurst exponent $H \in (0, 1)$ or equivalently the degree of persistence, d, where $d = H - \frac{1}{2}$. This parameter is often referred to as the parameter of long range dependence or long memory. Given any $d \in (-\frac{1}{2}, \frac{1}{2})$, the associated fBm, W^d, is a Gaussian stochastic process that can be defined on an appropriate probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $W^d(0) = 0$, with $\mathbb{E}\left(W^d(t)\right) = 0$

[1]Note that when κ is zero, the predicted yields from the classical Vasicek model will be $I(1)$. A general result, which we support in section 8, is that κ is generally found to be small but significantly larger than zero. This implies that predicted yields are $I(0)$ even though it is difficult to distinguish the observed yields from I(1) processes using unit root type tests or tests for stationary vs. non stationary long memory.

for all $t \geq 0$ and covariance

$$\mathbb{E}\left[W^d(t)W^d(s)\right] = \frac{\sigma^2}{2}\left[t^{2d+1} + s^{2d+1} - |t-s|^{2d+1}\right], \quad t \geq s.$$

Here, $\mathbb{E}$ denotes expectation with respect to the probability measure $\mathbb{P}$. From the covariance it is obvious that W^d exhibits long memory for $d > 0$, in the sense that

$$\sum_{n=1}^{\infty} \mathbb{E}\left[W^d(1)\left[W^d(n+1) - W^d(n)\right]\right] = \frac{\sigma^2}{2}\sum_{n=1}^{\infty}\left[(n+1)^{2d+1} - 2n^{2d+1} + (n-1)^{2d+1}\right] = \infty.$$

When $d < 0$ the fBm exhibits intermediate or anti persistent memory. Note that when $d = 0$, the correlation between increments is zero and we recover the standard Brownian motion with variance σ^2. Thus, the correlations of the fBm extend to arbitrary long periods, independent of time, having a strong effect on the evolution of the fBm.

In view of the usefulness of the fBm in mathematical finance, the recent focus has been on developing stochastic calculus for the fBm, see Duncan et al. (2000) and Hu & Øksendal (2003). However, for any bounded function $f(t, \eta)$ (which will be defined later on) and $d \in (0, \frac{1}{2})$, Biagini, Øksendal, Sulem & Wallner (2004) showed that the integral which is defined pathwise with respect to the fBm,

$$\int_a^b f(t,\eta)dW^d(t) \equiv \lim_{|\Delta t_k| \to 0} \sum_k f(t_k, \eta)\left(W^d(t_k+1) - W^d(t_k)\right),$$

leads to arbitrage in financial applications. This is an artifact of the property that the quadratic variation of $W^d(t)$ over $[0, 1]$ is zero for $0 < d < \frac{1}{2}$, see Novaisa (2000). However, an alternative integration theory based on the so-called Wick product[2] $\diamondsuit$ was introduced by Duncan et al. (2000),

$$\int_a^b f(t,\eta)dW^d(t) \equiv \lim_{|\Delta t_k| \to 0} \sum_k f(t_k, \eta) \diamondsuit \left(W^d(t_k+1) - W^d(t_k)\right). \qquad (1)$$

These types of integrals are named fractional Itô integrals, because they share many of the properties of the classical Itô integrals (where the integrator is the usual Brownian motion). For instance, contrary to the pathwise definition, the fractional Itô integral implies that

$$\mathbb{E}\left[\int_R f(t,\eta)dW^d(t)\right] = 0,$$

and it therefore shares this property with the standard Itô integral.

[2]See Hida, Kuo, Potthoff & Streit (1993) for details on Wick calculus.

Using the definition in (1), Duncan et al. (2000) showed that if $f(x)$ is a twice differentiable function, then

$$f(W^d(t)) = f(0) + \int_0^T f'(W^d(t))dW^d(s) + (d + \frac{1}{2}) \int_0^T s^{2d} f''(W^d(t))ds. \quad (2)$$

Formula (2) can be expanded to cover more general situations. For instance, let $\eta(t) = \int_0^t a(u)dW^d(u)$, where $a(t)$ belongs to the Hilbert space L^2_φ.[3] Then

$$\begin{aligned} f(t, \eta(t)) &= f(0,0) + \int_0^t \frac{\partial f}{\partial s}(s, \eta(s))\ ds + \int_0^t \frac{\partial f}{\partial \eta}(s, \eta(s))\, a(s)dW^d(s) \\ &\quad + \int_0^t \frac{\partial^2 f}{\partial \eta^2}(s, \eta(s)) \int_0^s a(u)\varphi(s,u)\ du\ ds. \end{aligned} \quad (3)$$

Benth (2003) and Hu & Øksendal (2003) proved that this kind of fractional Itô calculus leads to absence of arbitrage in a fractional Black-Scholes market. In the following section we will explain this in more detail and how we apply this result in our setting.

3 The Quasi-Conditional Expectation, Quasi-Martingales and Absence of Arbitrage

Recent developments in fractional white noise calculus (or fractional Itô calculus) by Duncan et al. (2000) and Hu & Øksendal (2003) and the extensions concerning quasi-conditional expectations by Benth (2003) allow us to introduce an arbitrage free setup for bond prices governed by fractional Brownian motions. In this section, we review the concepts of quasi-conditional expectations and quasi-martingales, and use their properties to derive arbitrage free dynamics for the bond price.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space supporting a fractional Brownian motion, $W^d(t)$, with long memory parameter $0 < d < 1/2$, where $\mathcal{F}_t^d$ is the σ-field generated by $\{W^d(t) : s \leq t\}$, and introduce the weight function

$$\varphi(s,t) = 2d(d+1/2)|s-t|^{2d-1}, \quad s, t \in \mathbb{R}.$$

[3] The Hilbert space L^2_φ is defined as the set of functions $f : \mathbb{R}^n \longmapsto \mathbb{R}$ such that

$$\begin{aligned} |f|^2_{\varphi,n} &:= \langle f, f \rangle_{\varphi,n} \\ &= \int\limits_{\mathbb{R}^n \times \mathbb{R}^n} f(s_1, \ldots, s_n)\, f(t_1, \ldots, t_n)\, \varphi(s_1,t_1) \ldots \varphi(s_n,t_n) \mathrm{d}s\mathrm{d}t < \infty. \end{aligned}$$

Now for a symmetric function $f \in L^2_\varphi$, we can define the iterated fractional Wiener integral (see Benth (2003))

$$I_n(f) = \int\limits_{\mathbb{R}^n} f \mathrm{d}W^d_{\otimes n} = n! \int\limits_{s_1<\ldots<s_n} f\left(s_1, \ldots, s_n\right) \mathrm{d}W^d(s_1) \ldots \mathrm{d}W^d(s_n).$$

From Benth (2003, Lemma 2.2) this definition allows us to write the fractional Itô integral (1) as

$$\begin{aligned}\int_0^t f(s,\eta) dW^d(s) &\equiv \lim_{|\Delta s_k|\to 0} \sum_k f(s_k,\eta) \diamondsuit \left(W^d(s_k+1) - W^d(s_k)\right) \\ &= \sum_{n=0}^{\infty} I_{n+1}\left(\frac{1}{n-1} f \mathbb{I}^{\otimes(n+1)}_{[0,t)}\right),\end{aligned}$$

and state the fractional Itô formula in (3), see Duncan et al. (2000), Hu & Øksendal (2003), and Benth (2003) for details.

Now if we assume that y has the expansion $y = \Sigma_{n=0}^{\infty} I_n(f)$ and belongs to $L^2(\Omega, \mathcal{F}, \mathbb{P})$, the quasi-conditional expectation of y with respect to $\mathcal{F}^d_t$ is

$$\widehat{\mathbb{E}}[y|\mathcal{F}^d_t] = \sum_{n=0}^{\infty} I_n\left(f \mathbb{I}^{\otimes n}_{[0,t]}\right),$$

where $\widehat{\mathbb{E}}[y|\mathcal{F}^d_0] = \mathbb{E}[y]$, and if y is $\mathcal{F}^d_t$-adapted, then $\widehat{\mathbb{E}}[y|\mathcal{F}^d_t] = y$. Note that the quasi-conditional expectation operator satisfy the classical law of iterated (double) expectations, $\widehat{\mathbb{E}}\left[\widehat{\mathbb{E}}[y|\mathcal{F}^d_t]\right] = \mathbb{E}[y]$. From the above definition, Hu, Øksendal & Salopek (2005) introduced the $\mathcal{F}^d_t$-adapted stochastic process $M(t) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and named it a quasi-martingale assuming that $\widehat{\mathbb{E}}[M(t)|\mathcal{F}^d_s] = M(s)$ for every $0 \leq s \leq t \leq \infty$, and Benth (2003) proved that

$$M(t) = M(0) + \int_0^t N(s) \mathrm{d}W^d(s),$$

where $N(t)$ is defined by its expansion, see Benth (2003) for details. From this quasi-martingale representation theorem, it is easy to see that the fractional Brownian motion is a quasi-martingale. Furthermore, the stochastic exponential with $f \in L^2_\varphi(\mathbb{R})$,

$$\varepsilon(t) = \exp\left(\int_0^t f(s) \mathrm{d}W^d(s) - \frac{1}{2}\left|f \mathbb{I}_{[0,t]}\right|^2_\varphi\right),$$

is a quasi-martingale, and generally any stochastic process, $x(t) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, is a quasi-martingale if there exist $\mathcal{F}^d_t$-adapted stochastic processes $\varepsilon(t)$ and $a(t)$

such that

$$\begin{aligned} x(t) &= x(0) + \int_0^t \varepsilon(s)\mathrm{d}s + \int_0^t a(s)\mathrm{d}W^d(s) \\ &= x(0) + \int_0^t \varepsilon(s)\mathrm{d}s + \eta(t), \end{aligned}$$

where $\mathbb{E}\left[\int_0^T \int_0^T a(s)a(t)\varphi(s,t)\mathrm{d}s\mathrm{d}t\right] < \infty$, and $\eta(t)$ is defined in (3). Hence $f\left(t, W^d(t)\right)$ becomes a quasi-martingale by appealing to the fractional Itô formula.

To derive arbitrage free dynamics of non defaultable bond prices, we need to apply the fractional version of the Girsanov transform

$$W^{\mathbb{Q},d}(t) = W^d(t) + \int_0^t \lambda(s)ds,$$

where the fractional Brownian motion under $\mathbb{P}$, $W^d(t)$, is transformed into the fractional Brownian motion, $W^{\mathbb{Q},d}(t)$, under the risk neutral probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}_t^d)$, which is equivalent to $\mathbb{P}$, see Benth (2003) for details.

If we define the stochastic process $x(t) \equiv x(t, \eta(t))$, where $\eta(t) = \int_0^t a(u)dW^d(u)$, and define the price of the bond $P(t) \equiv P(t, x(t))$, then we can write the dynamics of $dP(t)$ under the probability measure $\mathbb{P}$ by applying the fractional Itô formula in (3)

$$\begin{aligned} dP(t) &= \left(\frac{\partial P}{\partial t} + \frac{\partial P}{\partial x}\frac{\partial x}{\partial t}\right) dt + \frac{\partial P}{\partial x}\frac{\partial x}{\partial \eta}a(t)dW^d(t) \\ &\quad + \frac{\partial^2 P}{\partial x^2}\left(\frac{\partial x}{\partial \eta}\right)^2 \left(\int_0^s a(u)\varphi(s,u)du\right) dt \\ &= \left(\frac{\partial P}{\partial t} + \frac{\partial P}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial^2 P}{\partial x^2}\left(\frac{\partial x}{\partial \eta}\right)^2 \left(\int_0^s a(u)\varphi(s,u)du\right)\right) dt + \frac{\partial P}{\partial x}\frac{\partial x}{\partial \eta}a(t)dW_t^d \\ &\equiv \mu(t,P)dt + \sigma(t,P)dW^d(t), \end{aligned}$$

see Theorem 1 below.

To find the conditions under which we have absence of arbitrage, we follow the classical numeraire inversion theorem (see e.g. Duffie (1996)), which implies that the no arbitrage property of $P(t)$ will hold if, and only if, it holds for the normalized process, $Z(t) = P(t)/B(t)$, where $B(t) = \exp\left(\int_0^t r(s)ds\right)$ is the risk free money market/bank account. Using the fractional Itô formula we find

$$\frac{dZ(t)}{Z(t)} = \frac{dP(t)}{P(t)} - \frac{dB(t)}{B(t)} = \left(\frac{\mu(t,P)}{P(t)} - r(t)\right) dt + \frac{\sigma(t,P)}{P(t)}dW^d(t),$$

and applying the fractional Girsanov transform we find

$$\frac{dZ(t)}{Z(t)} = \left(\frac{\mu(t,P) - \sigma(t,P)\lambda(t)}{P(t)} - r(t)\right) dt + \frac{\sigma(t,P)}{P(t)}dW^{\mathbb{Q},d}(t). \tag{4}$$

Since $dZ(t)/Z(t) = \sigma(t,P)P(t)^{-1}W^{\mathbb{Q},d}(t)$, this is a quasi-martingale under the measure $\mathbb{Q}$, if we define $\lambda(t) = \mu(t,P)/\sigma(t,P)$, and if $\sigma(t,P)P(t)^{-1}$ is bounded in the sense that

$$\mathbb{E}\left[\int_0^T \int_0^T \sigma(s,P)P(s)^{-1}\sigma(t,P)P(t)^{-1}\varphi(s,t)\mathrm{d}s\mathrm{d}t\right] < \infty.$$

This relation reveals the arbitrage free dynamics of $dP(t)$

$$dP(t) = r(t)P(t)dt + \sigma(t,P)dW^{\mathbb{Q},d}(t).$$

Combining this with applying the fractional Itô formula directly on $P(t)$ and using the fractional Girsanov transform, i.e.

$$dP(t) = \left(\mu(t,P) - \frac{\partial P}{\partial x}\frac{\partial x}{\partial \eta}a(t)\lambda(t)\right)dt + \sigma(t,P)dW^{\mathbb{Q},d}(t),$$

we find the general condition that

$$\mu(t,P) - \frac{\partial P}{\partial x}\frac{\partial x}{\partial \eta}a(t)\lambda(t) - r(t)P(t) = 0. \tag{5}$$

This is the fractional version of the fundamental bond pricing equation.

A less mathematical and more economically minded approach to understanding the no arbitrage conditions is based on the concept of a self financing portfolio strategy. A portfolio strategy $\{\theta(t)P(t)\}$ is said to be self financing if its value changes only due to changes in the asset prices. Stated differently, except for the final withdrawal, no additional cash inflows or outflows occur after the initial time $t = 0$. This can be expressed as

$$\theta(t)P(t) = \theta(0)P(0) + \int_0^t \theta(s)dP(s),$$

where $P(t)$ is the fractional Itô price process (the bond price process) and $\theta(0)P(0)$ is the initial value of the portfolio. The self financing portfolio strategy will be a quasi-martingale after discounting under the risk neutral probability measure $\mathbb{Q}$, which is a well known result from the case of the standard Brownian motion.

From Benth (2003, Lemma 4.1) we know that if $\{\theta(t)P(t)\}$ is self financing, then

$$\begin{aligned}\theta(t)P(t)/B(t) &= \theta(0)P(0) + \int_0^t \theta(s)d(P(s)/B(s)),\\ \theta(t)Z(t) &= \theta(0)Z(0) + \int_0^t \theta(s)dZ(s),\end{aligned} \tag{6}$$

where $B(t)$ is defined above, and by the quasi-martingale representation theorem the right hand side of (6) is a fractional Itô integral under $\mathbb{Q}$ since $Z(s) =$

$P(s)/B(s)$ is a quasi-martingale (see equation (4)). This result parallels the theory for standard Brownian motions. Now, the portfolio strategy is said to be an arbitrage if the value at time $t = 0$ is non positive while the closing value at time T is non negative with positive probability. In fact this entails that the cash inflows from the strategy are non negative in all states of nature.

Thus, the arbitrage free price of the non defaultable bond at time t is

$$
\begin{aligned}
P(t) &= B(t)\widehat{\mathbb{E}}_{\mathbb{Q}}[P(T)/B(T)|\mathcal{F}_t^d] = \widehat{\mathbb{E}}_{\mathbb{Q}}[(B(t)/B(T))P(T)|\mathcal{F}_t^d] \\
&= \widehat{\mathbb{E}}_{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)ds\right)P(T)|\mathcal{F}_t^d\right] \\
&= \widehat{\mathbb{E}}_{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)ds\right)|\mathcal{F}_t^d\right], \qquad (7)
\end{aligned}
$$

since $P(T) = 1$, where $\widehat{\mathbb{E}}_{\mathbb{Q}}$ is the quasi-conditional expectation under $\mathbb{Q}$. Again the parallel to the theory for standard Brownian motions is evident. From a computational point of view equation (7) reveals that we can price a non defaultable bond by taking the expected value of the discounted payoff, where the expectations are taken under the measure $\mathbb{Q}$, and the discounting is made at the risk free rate. This approach is equivalent to solving the fundamental bond pricing equation in (5). We explore this in more detail in the next sections.

Before turning to the explicit derivation of the dynamics of the bond price in the fractional setup, we note that Björk & Hult (2005) have criticized Hu & Øksendal (2003)'s proposed solution to the problem of the admittance of arbitrage in a fractional Black-Scholes market. Under the specific model and framework of Hu & Øksendal (2003), Björk & Hult (2005) show that the methodology can lead to a portfolio with a positive probability of a negative value, when the fractional Brownian motion takes values in the range (1/2,3/2), and that the value of the portfolio depends on time. This implies that one must know the entire path of the portfolio components. Justified by this, Björk & Hult (2005) conclude that the results are not economically meaningful. Note, however, that in order to prove that the value of the portfolio can become negative, Björk & Hult (2005) have to choose the number of risky assets as a linear function of the price of the risky asset itself. Otherwise the critique does not hold. This is of course sensible, but far from practice since traders often delta and/or vega hedge, which leads to far more complex portfolio value dynamics. Together with the fact that Björk & Hult (2005)'s critique is heavily based on the fractional Black-Scholes setup and not directly applicable to our term structure model, we believe that our setup excludes arbitrage possibilities based on the arguments of Benth (2003). One point of Björk & Hult (2005) that carries through to our framework is that the price of the bond depends more explicitly on time compared to the classical framework, since the increments of the fractional Brownian motion is long-range dependent. This is a standard feature of long-range dependent processes, which of course also affects our model. We do not think this is a critical problem but just implies that one has to think of portfolio values in a slightly different manner than usual.

4 The Fractional Ornstein-Uhlenbeck Process

In continuous time the analogy to the discrete $AR(1)$ is the Ornstein-Uhlenbeck process (OUP). Recall the Gaussian OUP

$$dx(t) = \kappa\,(\theta - x(t))\,dt + \sigma dW(t) \tag{8}$$

or

$$x(t) = x(0) + \int_0^t \kappa\,(\theta - x(u))\ \ du + \sigma W(t),$$

where $\{W(t) : t \geq 0\}$ is a standard Brownian motion.

The natural extension to a long memory setting of this model is to apply a fractional Brownian motion (fBm) and write the first order stochastic differential equation (SDE) in (8) with respect to a fractional Gaussian noise

$$dy(t) = \kappa\,(\theta - y(t))\,dt + \sigma dW^d(t), \tag{9}$$

where $\left\{W^d(t) : t \geq 0\right\}$ is the fractional Brownian motion.

This SDE is the continuous time analogy of a discrete $ARFIMA(1, d, 0)$ process, and it is intuitively named the fractional Ornstein-Uhlenbeck process (FOUP). Note that when $d = 0$ model (8) coincides with model (9) given $x \equiv y$.

In fact, it is known from Comte & Renault (1996, Proposition 9) that in the general case of model (9), the d-derivative $y^{(d)}(t)$ is governed[4] by the usual OUP

$$dy^{(d)}(t) = \kappa\left(\theta - y^{(d)}(t)\right)dt + \sigma dW(t). \tag{10}$$

In the following, eq. (9) will be used as an example of a fractional model for the instantaneous interest rate. First we consider a general setup.

5 A Diffusion Model for the Instantaneous Interest Rate

In the following, we base our derivations on a simple one-factor model. This is done to enhance the exposition and to ease the introduction and understanding of the fractional model. Note that the setup is readily extendible to involve multiple factors, which could be correlated.

[4]The discrete time analogues of the OUP and FOUP processes are related via $y^{(d)}(t) = (1 - L)^d y(t)$.

In our setup for a one factor model, we assume the instantaneous interest rate $r(t)$ follows an SDE with solution given by

$$r(t) = m(t) + g(t)\,\eta(t),$$

where $\eta(t)$ is given by $\mathrm{d}\eta(t) = \sigma(t)\mathrm{d}W(t)$, $W(t)$ a Brownian motion. We assume that $m(\cdot)$ and $g(\cdot)$ are differentiable.

From Itô's formula we know that the SDE for the instantaneous interest rate is

$$\mathrm{d}r(t) = [m'(t) + g'(t)\,\eta(t)]\,\mathrm{d}t + g(t)\sigma(t)\mathrm{d}W(t).$$

Note that this is the usual approach turned upside down: Here we start with the solution. The reason being that this is the way we generalize from the usual term structure model to a fractional version.

We are considering a classical one factor term structure model, so the bond prices, which we denote $P(t,\tau)$, end up being functions of $r(t)$. Here $\tau = T - t$ is the time to maturity, T is the maturing time. In the sequel we omit τ, as we write $P(t)$ for $P(t,\tau)$.

By making the no arbitrage assumption, we obtain a well known second order partial differential equation (PDE) for $P(t)$, thereby finding the bond pricing formula. Since the methodology in the following is to be used also in the fractional case, we do this in some detail.

The bond price $P(t)$ is a function of t and $\eta(t)$ in the following way. Let $f(t,\eta) = m(t) + g(t)\eta$, then $r(t) = f(t,\eta(t))$. The bond price is a function of $r(t)$, and therefore a function of $\eta(t)$. We write it as some function $c(\cdot,\cdot)$

$$P(t) = c(t, f(t,\eta))\ ^{5}.$$

According to Itô's formula we need the following derivatives

$$\begin{aligned}
P_t(t,\eta) &= \frac{\partial P}{\partial t} + \frac{\partial P}{\partial f}\cdot\frac{\partial f}{\partial t} = \frac{\partial P}{\partial t} + \frac{\partial P}{\partial f}\cdot(m'(t) + g'(t)\,\eta(t))\\
P_\eta(t,\eta) &= \frac{\partial P}{\partial f}\cdot\frac{\partial f}{\partial \eta} = \frac{\partial P}{\partial f}\cdot g(t)\\
P_{\eta\eta}(t,\eta) &= \frac{\partial^2 P}{\partial f^2}\cdot\left(\frac{\partial f}{\partial \eta}\right)^2 = \frac{\partial^2 P}{\partial f^2}\cdot g(t)^2.
\end{aligned}$$

Again, according to Itô's formula we get

$$\begin{aligned}
\mathrm{d}P(t) &= \left[\frac{\partial P}{\partial t} + \frac{\partial P}{\partial f}\,(m'(t) + g'(t)\,\eta(t)) + \frac{1}{2}\frac{\partial^2 P}{\partial f^2}g(t)^2\sigma(t)^2\right]\mathrm{d}t\\
&\quad + \frac{\partial P}{\partial f}g(t)\sigma(t)\mathrm{d}W(t).
\end{aligned}$$

[5] For example $c(t, x(t)) = \exp(a(\tau) + b(\tau)x(t))$.

Assuming $\mathrm{d}P(t) = \mu_P(t)P(t)\mathrm{d}t + \sigma_P(t)P(t)\mathrm{d}W(t)$, and assuming the no arbitrage condition $\mu_P(t) = r(t) + \lambda(r(t))\sigma_P(t)$, where $\lambda(r(t))$ is the market price of risk, we obtain the partial differential equation

$$\frac{1}{2}\frac{\partial^2 P}{\partial f^2} g(t)^2\sigma(t)^2 + \frac{\partial P}{\partial f}\left[m'(t) + g'(t)\,\eta(t) - \lambda(r(t))g(t)\sigma(t)\right] + \frac{\partial P}{\partial t} - P(t)r(t) = 0. \tag{11}$$

This is well known for example in the Vasicek model where we have

$$\begin{aligned}
m(t) &= x_0 K(t) + \kappa\theta K(t)\int_0^t K(s)^{-1}\mathrm{d}s, \text{ where } K(t) = \mathrm{e}^{-\kappa t}, \\
m'(t) &= -\kappa m(t) + \kappa\theta, \\
\sigma(t) &= \sigma K(t)^{-1}, \\
g(t) &= K(t), \\
g'(t) &= -\kappa K(t) \\
\lambda(r(t)) &= \lambda.
\end{aligned}$$

Inserting these into (11) reveals the classical bond pricing equation

$$\frac{1}{2}\frac{\partial^2 P}{\partial f^2}\sigma^2 + \frac{\partial P}{\partial f}\left[\kappa(\theta - r(t)) - \lambda\sigma\right] + \frac{\partial P}{\partial t} - r(t)P(t) = 0,$$

which, assuming the exponential affine relation,

$$P(t) = \exp\left[a(\tau) + b(\tau)r(t)\right], \quad \tau = T - t,$$

leads to the so-called Ricatti equations

$$\begin{aligned}
a'(\tau) &= b(\tau)\left[\kappa\theta - \lambda\right] + \frac{1}{2}b(\tau)^2\sigma^2 \\
b'(\tau) &= -b(\tau)\kappa - 1,
\end{aligned}$$

with solutions

$$\begin{aligned}
a(\tau) &= -y_\infty\left[\tau + b(\tau)\right] - \frac{\sigma^2}{4\kappa}b(\tau)^2 \\
b(\tau) &= \frac{1}{\kappa}\left[e^{-\kappa\tau} - 1\right],
\end{aligned} \tag{12}$$

where $y_\infty = \theta - \lambda\sigma/\kappa - \frac{\sigma^2}{2\kappa^2}$.

To develop a fractional version we make use of the fractional Itô formula in eq. (3), which we generalize in the next section. In the sequel $r(t)$ is a completely different process (namely a fractionally integrated version) compared to the classical setup above, where $r(t)$ follows a continuous type AR(1).

What we do now is to replace the Brownian motion $W(t)$ with a fractional Brownian motion, named $W^d(t)$, and parameterized by the long memory parameter d.

As we have noted in sections 2 and 3, the process W^d, for $d \neq 0$, is neither a semimartingale nor a Markov process. Therefore standard techniques of stochastic calculus cannot be applied in a straightforward manner. However, as mentioned in sections 2 and 3, Duncan et al. (2000) have developed a stochastic calculus based on fractional Brownian motion that is analogous to the standard Itô calculus, which we investigate in the following.

6 The Fractional Itô Formula

Duncan et al. (2000, Theorem 4.3) have formulated and proved a fractional version of Itô's lemma. The result, which we reformulate below, generalizes Itô's lemma and states rules of calculus for stochastic differential equations generated by fractional Brownian motions. See also the work by Brody et al. (2002, Appendix).

Theorem 1 *(Duncan et al. (2000, Theorem 4.3))*
Let

$$\varphi(s,t) = 2d(d+1/2)|s-t|^{2d-1}, \quad \text{where } 0 < d < 1/2.$$

Given a deterministic function $\sigma(t)$ such that

$$\int_0^\infty \int_0^\infty \sigma(s)\sigma(t)\varphi(s,t)\,ds\,dt < \infty,$$

then the stochastic integral

$$\eta(t) = \int_0^t \sigma(s)\,dW^d(s)$$

is well defined for all $t \geq 0$.[6] *Furthermore given any twice continuously differentiable (in the second argument) function $G(\cdot,\cdot)$ with bounded derivatives, the following fractional Itô formula holds:*

$$dG\left[t,\eta(t)\right] = \left[\frac{\partial G}{\partial t} + \frac{\partial^2 G}{\partial \eta^2}\sigma(t)\int_0^t \varphi(s,u)\sigma(u)\,du\right]dt + \frac{\partial G}{\partial \eta}\sigma(t)\,dW^d(t).$$

[6]For example $W^d(t) = \int_0^t \mathrm{d}W^d(s)$, that is, $\sigma(s) = 1$.

6.1 Application of the Fractional Itô Formula

As noted earlier we consider the one-factor model in order to show the implications of introducing the fractional Brownian motion as the governing force in a simple and comprehensible manner. We are aware that for the model to be able to fully capture the rich dynamics of the term structure, we need to expand its complexity. However, the present section serves as an introduction to an extension of the classical setup, and thus is readily extendible to cover more complex models with multiple and possibly correlated factors.

We consider the integrated fractional Gaussian noise with $0 < d < 1/2$,

$$\eta(t) = \int_0^t \sigma(s)\mathrm{d}W^d(s), \text{ such that } \mathrm{d}\eta(t) = \sigma(t)\mathrm{d}W^d(t).$$

The fractionally integrated instantaneous interest rate process is given by $r(t) = f(t, \eta(t))$, where

$$f(t, \eta) = m(t) + g(t)\eta,$$

and $m(\cdot)$ and $g(\cdot)$ are differentiable functions. The bond price is assumed to be the function of t and $\eta(t)$ given in the fractional Itô formula

$$P(t) = P(t, f(t, \eta(t))) = G(t, \eta(t)).$$

Then we have

$$\begin{aligned} \mathrm{d}P(t) &= \left[\frac{\partial P}{\partial t} + \frac{\partial P}{\partial f}\left(m'(t) + g'(t)\,\eta(t)\right) + \frac{\partial^2 P}{\partial f^2} g(t)^2 \sigma(t) \int_0^t \varphi(s,u)\sigma(u)\mathrm{d}u\right] \mathrm{d}t \\ &\quad + \frac{\partial P}{\partial f} g(t)\sigma(t)\mathrm{d}W^d(t). \end{aligned}$$

The no arbitrage assumption leads to the fractional version of the fundamental bond pricing equation,

$$\frac{\partial^2 P}{\partial f^2}\, g(t)^2\sigma(t) \int_0^t \varphi(s,u)\sigma(u)\mathrm{d}u + \frac{\partial P}{\partial f}\left[m'(t) + g'(t)\,\eta(t) - \lambda\, g(t)\,\sigma(t)\right] + \frac{\partial P}{\partial t} - r(t)P(t) = 0.$$

Inserting the parameter values for the Vasicek model given in section 5, we obtain the PDE

$$\frac{\partial^2 P}{\partial f^2}\,\sigma^2\, K(t) \int_0^t \varphi(t,u)K(u)^{-1}\mathrm{d}u + \frac{\partial P}{\partial f}\left[\kappa(\theta - r(t) - \lambda\sigma\right] + \frac{\partial P}{\partial t} - r(t)P(t) = 0.$$

Inserting the derivatives of the bond price using the exponential affine form, and setting terms equal to zero, reveals the following fractional Ricatti equations

$$\begin{aligned} a'(\tau) &= b(\tau)^2\, K(t)\,\sigma^2 \int_0^t \varphi(t,u)K(u)^{-1}\mathrm{d}u \,+\, b(\tau)\left[\kappa\theta - \lambda\sigma\right] \\ b'(\tau) &= -1 - \kappa b(\tau). \end{aligned}$$

Obviously the solution for $b(\tau)$ is the same as before, and inserting this solution in the equation for $a(\tau)$ reveals

$$a'(\tau) = \left(\mathrm{e}^{-\kappa\tau} - 1\right)\left(\theta - \frac{\lambda}{\kappa}\sigma\right) + \frac{\sigma^2}{\kappa^2}\mathrm{e}^{-\kappa T}\left[\mathrm{e}^{-\kappa\tau} + \mathrm{e}^{\kappa\tau} - 2\right]\int_0^t \varphi(t,u)K(u)^{-1}\mathrm{d}u.$$

Integrating $a'(\tau)$, we can express

$$a(\tau) = -(\theta - \frac{\lambda\sigma}{\kappa})\,(\tau + b(\tau)) + \frac{\sigma^2}{\kappa^2}\mathrm{e}^{-\kappa T}\int_0^\tau H(s)\left[\mathrm{e}^{-\kappa s} + \mathrm{e}^{\kappa s} - 2\right]\mathrm{d}s, \qquad (13)$$

where

$$H(\tau) = 2d\,(d + \frac{1}{2})(T-\tau)^{2d}\int_0^1 \mathrm{e}^{\kappa(T-\tau)v}(1-v)^{2d-1}\mathrm{d}v,$$

and furthermore, integrating $b'(\tau)$ reveals

$$b(\tau) = \frac{1}{\kappa}\left[\mathrm{e}^{-\kappa\tau} - 1\right].$$

Now, in this fractional model, we write the exponential affine bond price

$$P(t) = \mathrm{e}^{a(\tau)+b(\tau)\,r(t)}, \quad \tau = T - t,$$

where the $a(\tau)$ function defined in (13) certainly is different from the $a(\tau)$ in equation (12), but the $b(\tau)$ function is unchanged. $b(\tau)$ just scales $r(t)$ which in contrast to the situation in section 5 is now driven by a long memory process. Note that $P(t)$ does indeed depend on past values of r, because $r(t)$ is fractionally integrated, see section 7.1 below. This implies that even if the $a(\tau)$ functions were identical, we would still obtain different bond prices (or yields). This fact is caused by the inclusion of the parameter of fractional integration d[7].

Another advantage of using the fractional version of the classical Vasicek model is that the realized bond return or yield variation exhibits long range dependence, which is a salient feature of the observed series, see Andersen & Benzoni (2005), Andersen, Bollerslev & Diebold (2005), and Nielsen & Frederiksen (2005). This implies that by expanding the complexity of the model by e.g. introducing more factors, it should be possible to alleviate any harsh critique of the affine model's ability to capture the volatility dynamics of the term structure. In the present paper we will not go into further details about this property but leave it for future research.

[7] Note that Comte & Renault (1996, proof of Proposition 12) derive a similar expression for the bond price. They base their derivation on the conditional expectation,

$$P(t,T) = \mathbb{E}_t^{\mathbb{Q}}[\exp(-\int_t^T r(s)\mathrm{d}s)|\mathcal{F}_t],$$

where $\mathcal{F}_t$ is the σ-field generated by the Brownian motion that governs $r(t)$, exploiting that the integral is Gaussian. We base our derivation directly on a fractional analogue of the classical setup.

7 The State Space Approach

In order to describe the evolution in zero coupon yields in a fractional setup, and estimate the implied parameters, and also to extract the unobservable state variable, we apply a state space formulation of the Vasicek model.

If the instantaneous interest rate process $r(t)$ follows an OUP, it is known that such a process can be written as the following transition

$$r(t) = \mathrm{e}^{-\kappa(t-s)} r(s) + \theta(1 - \mathrm{e}^{-\kappa(t-s)}) + \sigma \int_s^t \mathrm{e}^{-\kappa(t-u)} \mathrm{d}W(u),$$

where $W(\cdot)$ is a Brownian motion.

The exact state space formulation is based on the transition density, i.e. the conditional density of $r(t)$ given $r(s)$. It is known from Vasicek (1977) that the exact transition density is the product of normal densities, i.e. the equidistant state process $\{r(t + \Delta\, k)\}_{k \in Z}$ is a Markov process for any $\Delta > 0$.

In case $r(t)$ is an FOUP, then according to (10) the dth difference of $r(t)$, $r^{(d)}(t)$, may be written as

$$r^{(d)}(t) = \mathrm{e}^{-\kappa\Delta}\, r^{(d)}(s) + \theta(1 - \mathrm{e}^{-\kappa\Delta}) + \sigma \int_s^t \mathrm{e}^{-\kappa\Delta} \mathrm{d}W(u), \tag{14}$$

where $t - s = \Delta$. In other words, $r(t)$ sampled at equidistant time points is an ARFIMA$(1, d, 0)$ process. In this case $\{r(t + \Delta\, k)\}_{k \in Z}$ is not Markov, and to apply the Kalman filter in this case, we have to define the state variable (or vector) differently than in the ordinary case.

If we let $\mathbf{y}(t) = (y(1, t), \ldots, y(n, t))'$ be an n-dimensional vector of yields observed at time t, then the approach can be described as follows.

From the exponential affine relation, $P(t) = \exp\left[a(\tau) + b(\tau) r(t)\right]$, the yields can be expressed as

$$\mathbf{y}(t) = -\,\frac{1}{\tau} \log P(t) = \mathbf{d}(t) + \tilde{\mathbf{z}}(t)\, r(t), \tag{15}$$

where $\mathbf{d}(t)$ and $\tilde{\mathbf{z}}(t)$ are the n-dimensional vectors

$$\mathbf{d}(t) = \begin{pmatrix} -a(\tau_1)/\tau_1 \\ \vdots \\ -a(\tau_n)/\tau_n \end{pmatrix}, \quad \tilde{\mathbf{z}}(t) = \begin{pmatrix} -b(\tau_1)/\tau_1 \\ \vdots \\ -b(\tau_n)/\tau_n \end{pmatrix},$$

respectively. Adding an error term to (15), we obtain a measurement equation for the observed yields, where $r(t)$ is an unobserved variable.

7.1 A Kalman Filter for the ARFIMA$(1, d, 0)$

Now let $r(t)$ be an ARFIMA model according to (14). Then

$$r(t) = \mu + \varphi\, r(t-1) + \eta^{(d)}(t), \tag{16}$$

where $\mu = \theta(1-\mathrm{e}^{-\kappa\Delta})$, $\varphi = \mathrm{e}^{-\kappa\Delta}$, and $\eta^{(d)}(t)$ is integrated fractional white noise[8] with parameters d and σ^2. Then $\tilde{x}(t) = (1-L)^d r(t)$ is the AR(1) process with mean zero[9]

$$\tilde{x}(t) = \varphi\, \tilde{x}(t-1) + \tilde{\eta}(t),$$

where $\tilde{\eta}(t)$ is a white noise process with variance $\sigma^2(1-\mathrm{e}^{-2\kappa\Delta})/2\kappa$.

To obtain a feasible version, we truncate the infinite polynomial

$$(1-L)^d = \sum_{k=0}^{\infty} \left[\frac{\Gamma(k-d)}{\Gamma(k+1)\Gamma(-d)} \right] L^k$$

at the $(m-1)$th power, thereby defining the following finite filtered variable

$$x(t) = \sum_{k=0}^{m-1} c_k^{(d)}\, r(t-k), \tag{17}$$

where

$$c_k^{(d)} = \frac{\Gamma(k-d)}{\Gamma(k+1)\Gamma(-d)}, \qquad k = 0, 1, 2, \cdots, m-1.$$

Then, according to (16)

$$x(t) = \mu \sum_{k=0}^{m-1} c_k^{(d)} + \varphi x(t-1) + \eta(t). \tag{18}$$

Here $\eta(t) = \sum_{k=0}^{m-1} c_k^{(d)} \eta^{(d)}(t-k)$, which converges to a white noise process for $m \to \infty$.

Consider the m-dimensional vector $\mathbf{x}(t)$ defined from equation (18) by

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ x(t-1) \\ x(t-2) \\ \vdots \\ x(t-m+1) \end{bmatrix}.$$

[8] Fractional white noise is defined as a stationary ARFIMA$(0, d, 0)$ process. Note that stationarity requires $-\frac{1}{2} < d < \frac{1}{2}$.

[9] Note that if $0 < d \leq 1$, then the mean $(1-L)^d\, \mu = 0$ because $\sum_{k=0}^{\infty} c_k^{(d)} = 0$.

Then the following transition holds

$$\mathbf{x}(t) \;\; = \;\; \boldsymbol{\mu} + \mathbf{T}\,\mathbf{x}(t-1) + \boldsymbol{\eta}(t),$$

where $\boldsymbol{\mu}$, $\boldsymbol{\eta}(t)$ and $\mathbf{T}$ are the two $(m \times 1)$ vectors and partitioned $(m \times m)$ matrix given as

$$\boldsymbol{\mu} = \begin{bmatrix} \mu \sum_{k=0}^{m-1} c_k^{(d)} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \boldsymbol{\eta}(t) = \begin{bmatrix} \eta(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \text{ and } \mathbf{T} = \begin{bmatrix} \varphi & \mathbf{0}'_{m-1} \\ \mathbf{I}_{m-1} & \mathbf{0}_{m-1}. \end{bmatrix}.$$

To be able to express the vector of observed yields as a function of the state vector $\mathbf{x}(t)$, we derive from (17)

$$\mathbf{x}(t) = \mathbf{A}_1\,\mathbf{r}(t) + \mathbf{A}_2\,\mathbf{r}(t-m),$$

where

$$\mathbf{r}(t) = \begin{bmatrix} r(t) \\ r(t-1) \\ \vdots \\ r(t-m+1) \end{bmatrix}, \qquad \text{and}$$

$$\mathbf{A}_1 = \begin{bmatrix} 1 & c_1^{(d)} & c_2^{(d)} & \cdots & c_{m-1}^{(d)} \\ 0 & 1 & c_1^{(d)} & \cdots & c_{m-2}^{(d)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ c_{m-1}^{(d)} & 0 & 0 & \cdots & 0 \\ c_{m-2}^{(d)} & c_{m-1}^{(d)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_1^{(d)} & c_2^{(d)} & \cdots & c_{m-1}^{(d)} & 0 \end{bmatrix}.$$

Since $\mathbf{A}_1$ is non singular with inverse

$$\mathbf{A}_1^{-1} = \begin{bmatrix} 1 & c_1^{(-d)} & c_2^{(-d)} & \cdots & c_{m-1}^{(-d)} \\ 0 & 1 & c_1^{(-d)} & \cdots & c_{m-2}^{(-d)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

where

$$c_k^{(-d)} = \frac{\Gamma(k+d)}{\Gamma(k+1)\Gamma(d)}, \qquad k = 0, 1, 2, \cdots, m-1,$$

we obtain that

$$\mathbf{r}(t) = \mathbf{A}_1^{-1}\mathbf{x}(t) - \mathbf{A}_1^{-1}\mathbf{A}_2\mathbf{r}(t-m),$$

and the first row in this equation is

$$\begin{aligned} r(t) &= \sum_{k=0}^{m-1} c_k^{(-d)}\, x(t-k) - \sum_{k=1}^{m-1} \left(\sum_{j=k}^{m-1} c_j^{(-d)} c_{m+k-j-1}^{(d)} \right) r(t-m-k) \quad (19) \\ &\approx \sum_{k=0}^{m-1} c_k^{(-d)}\, x(t-k). \end{aligned}$$

The second term in (19) is an artifact of the truncation of $\tilde{x}(t)$ induced by the finite filter defining $x(t)$ in (17), and we omit it since if $m \to \infty$ this term vanishes.

Returning to the relation in (15), we obtain the following

$$\begin{aligned} \mathbf{y}(t) &= \mathbf{d}(t) + \tilde{\mathbf{z}}(t)\, r(t) \\ &\approx \mathbf{d}(t) + \tilde{\mathbf{z}}(t) \left(\sum_{k=0}^{m-1} c_k^{(-d)}\, x(t-k) \right) \\ &= \mathbf{Z}(t)\, \mathbf{x}(t) + \mathbf{d}(t), \end{aligned} \tag{20}$$

where[10]

$$\mathbf{Z}(t) = \tilde{\mathbf{z}}(t) \begin{bmatrix} 1 & c_1^{(-d)} & \cdots & c_{m-1}^{(-d)} \end{bmatrix}.$$

Having defined the state space equations above, we have the usual Kalman filter techniques available, see below in section 7.2.

7.2 The Econometric Procedure

Given the linear relation between the observed yields and the unobserved state vector in (20), we can derive the conditional density $\mathbf{y}(t) \mid \mathbf{x}(t)$. Furthermore, we write the measurement equation corresponding to (20) for the observed yields as

$$\mathbf{y}(t) = \mathbf{d}^{(\boldsymbol{\psi})}(t) + \mathbf{Z}^{(\boldsymbol{\psi})}(t)\mathbf{x}(t) + \boldsymbol{\varepsilon}(t), \qquad \boldsymbol{\varepsilon}(t) \sim \mathbf{NID}\left(\mathbf{0}, \mathbf{H}^{(\psi)}\right), \tag{21}$$

where $\boldsymbol{\psi} = (d, \kappa, \theta, \sigma, \lambda)'$ is the vector of parameters. Equation (21) includes an error term $\boldsymbol{\varepsilon}(t)$ whereas the corresponding equation (15) of the affine yields does not include any errors. This is actually not inconsistent with the applied exponential affine term structure model. Suppose the fractional Vasicek model is the true mechanism governing prices and yields, then the errors just account for the possibility of bid ask spreads, non simultaneity of observations, errors in the data, etc. The size of such errors is expected to be small compared to the variation in yields.

For a Gaussian state space model, the Kalman filter provides an optimal solution to prediction, updating and evaluating the likelihood function. The

[10] Note that $\tilde{\mathbf{z}}(t)$ is $(n \times 1)$, whereas $\mathbf{Z}(t)$ is $(n \times m)$.

Kalman filter recursion is in its simplicity just a set of equations, which allows an estimator to be updated once a new observation becomes available. The Kalman filter first forms an optimal predictor of the unobservable state variable vector given previously estimated values. This prediction is obtained using the conditional distribution of the unobserved state variable, and is updated using the information provided by the observed variables. The implied prediction errors are then used to evaluate the likelihood function cf. Harvey (1989).

In deriving the log likelihood function to be maximized, we need the transition equation, which is the exact discrete time distribution of the state vector $\mathbf{x}(t)$,

$$\mathbf{x}(t) = \boldsymbol{\mu} + \mathbf{T}\mathbf{x}(t-1) + \boldsymbol{\eta}(t), \qquad \boldsymbol{\eta}(t) \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{Q}),$$

where the system matrices $\boldsymbol{\mu}$, $\mathbf{T}$ and $\mathbf{Q}$, are functions of the parameters in the stochastic process for $\mathbf{x}(t)$. The measurement equation (21) is linear in the state variable and a standard Kalman filter method can by applied. This leads to the following log likelihood function

$$\log L = \sum_{t=1}^{T} -\frac{n}{2}\log(2\pi) - \frac{1}{2}\log|\mathbf{F}_t| - \frac{1}{2}\boldsymbol{\nu}_t'\mathbf{F}_t^{-1}\boldsymbol{\nu}_t, \tag{22}$$

where $\boldsymbol{\nu}_t$ is the n column vector of prediction errors with covariance matrix $\mathbf{F}_t$.

The extraction of the parameters through (22) evolves in two steps. First, the prediction step (in the following we suppress the dependence on the parameter vector $\boldsymbol{\psi}$ for convenience)

$$\hat{\mathbf{x}}_{t|t-1} = \mathbf{T}\,\hat{\mathbf{x}}(t-1) + \boldsymbol{\mu}. \tag{23}$$

When $\mathbf{y}(t)$ is observed, an updated estimate of (23) is computed, such that the fit to the observed yields for the current date is optimal in the mean squared error sense, where the mean square error (MSE) matrix is

$$\boldsymbol{\Sigma}_{t|t-1} = \mathbf{T}\Sigma_{t-1}\mathbf{T}' + \sigma^2\mathbf{Q}.$$

Here $\boldsymbol{\Sigma}_{t|t-1}$ is the prediction of the covariance matrix of the estimation error of the state variable $\mathbf{x}(t)$, i.e. $\boldsymbol{\Sigma}_{t|t-1}$ is the prediction of the covariance matrix of $\mathbf{x}(t) - \hat{\mathbf{x}}(t)$, and

$$\begin{aligned}
\Sigma_t &= \boldsymbol{\Sigma}_{t|t-1} - \boldsymbol{\Sigma}_{t|t-1}\mathbf{Z}(t)'\mathbf{F}_t^{-1}\mathbf{Z}(t)\boldsymbol{\Sigma}_{t|t-1} \\
&= (I - \boldsymbol{\Sigma}_{t|t-1}\mathbf{Z}(t)'\mathbf{F}_t^{-1}\mathbf{Z}(t))\boldsymbol{\Sigma}_{t|t-1} \\
&= (\mathbf{I} - \mathbf{K}(t)\mathbf{Z}(t))\boldsymbol{\Sigma}_{t|t-1}.
\end{aligned}$$

Hence, the additional information contained in $\mathbf{y}(t)$ is used to obtain a more precise estimator of $\mathbf{x}(t)$, namely

$$\hat{\mathbf{x}}(t) = \hat{\mathbf{x}}_{t|t-1} + \boldsymbol{\Sigma}_{t|t-1}\mathbf{Z}(t)'\mathbf{F}_t^{-1}\boldsymbol{\nu}(t),$$

where

$$\begin{aligned}\boldsymbol{\nu}_t &= \mathbf{y}(t) - \widehat{\mathbf{y}}(t) = \mathbf{y}(t) - \big(\mathbf{d}(t) + \mathbf{Z}(t)\hat{\mathbf{x}}_{t|t-1}\big) \\ \mathbf{F}_t &= \mathbf{Z}(t)\boldsymbol{\Sigma}_{t|t-1}\mathbf{Z}(t)' + \mathbf{H}.\end{aligned}$$

8 The US Term Structure of Interest Rates

In this empirical illustration, we model the US term structure of interest rates. We propose a one factor model where the state variable evolves according to a fractional Ornstein-Uhlenbeck process, and propose estimating the implied parameters using the approach described above.

8.1 Data Description

We extracted weekly observations on US Treasury strips from the Bloomberg database, which covers interest rates with 1,2,4,6,8 and 10 years of duration in the period February 1991 to January 2002, a total of $n = 558$ observations.

To give a brief insight into the properties of the data, Table 1 shows descriptive statistics including p-values for the Jarque-Bera normality test and long memory estimates using the Gaussian semiparametric estimator of Robinson (1995).

Insert Table 1 about here

A pronounced feature of the data is that the shorter rates are more volatile, and the fact that the mean increases with maturity indicates that the instantaneous interest rate has a long term mean less than 4.9%. More interestingly, we find that the kurtosis is fairly close to 3. The negligible skewness for the intermediate rates therefore implies that normality cannot be rejected. This is very convenient since we let the state variable be described by the Gaussian FOUP, which in the affine class of term structure models implies that the predicted yields are normal.

Recall that in our model setup, we assume that the instantaneous interest rate is governed by an fBm with $0 < d < 1/2$. Since we assume that the (predicted) bond yields are linear in the instantaneous interest rate, $r(t)$, we know that the long memory property of $r(t)$ should project itself to the yields, see e.g. Dittmann & Granger (2002) and the references therein. As a consequence, the observed bond yields are assumed to be $I(d)$. To evaluate this assumption, Table 1 also presents long memory estimates, $\tilde{d}$, of the observed yields based on the Gaussian semiparametric estimator (GSE) of Robinson (1995)[11]. Note that the estimates

[11]In the frequency domain the GSE estimates $\tilde{d}$ such that $(1-L)^{\tilde{d}}y(t)$ is a white noise.

are close to one with the tendency that $\tilde{d} > 1$. These results are, however, not in conflict with our model assumptions. In our model the memory in the yields are parameterized through a combination of d and the mean reversion parameter κ, where a small κ will make the predicted yields behave like $I(1+d)$ processes. On the contrary, in the GSE, the memory in the yields enters the model only through $\tilde{d}$, and as a consequence, it seems natural that $\tilde{d} \approx 1 + d$ if the process slowly mean reverts.

8.2 The Empirical Results

Table 2 shows the results for the implied parameters of the two one factor models, i.e. the classical Vasicek model (OUP) and the fractional version (FOUP), and Table 3 shows their fit in terms of bias and root mean square error (RMSE).

Insert Tables 2-3 about here

The results for the ordinary one factor model is well known in the field of term structure modeling (see e.g. the references in the introduction). We confirm the finding that in order to capture the long maturity bond yields, the speed at which the instantaneous interest rate mean reverts is slow ($\kappa = 0.0158$). The very low κ is an artifact of including the longer yields in the analysis, and implies that the model predicts the yields of longer maturity bonds better. Nonetheless, the other estimates seem slightly of target. For instance, the result for θ implies that the long term mean is 5.29%, which is to high compared to the results in Table 1. Furthermore, the combination of the volatility parameter, $\sigma = 0.0109$, and the market price of risk parameter, $\lambda = -0.2700$, implies that the excess return on the bond with six years to maturity ($D = 6$) is $-\lambda\sigma D = 1.77\%$, which is too large if $\theta = 0.0529$.

Turning to the fractional Vasicek model, we see that incorporating the fBm matters. Even though the estimate of d is insignificantly larger than zero ($d = 0.0418$), this extension alleviates the deadlock of κ. Now, since the fBm helps capturing the dynamics of the long maturity bond yields, the instantaneous interest rate is allowed to mean revert ($\kappa = 0.2188$). Another plausible result of the fractional Vasicek model is that the long term mean is estimated to be 4.16%. Nonetheless, the increase in the market price of risk parameter ($\lambda = -0.6034$) renders to high an excess return of 4.53% on the bond with six years to maturity.

As noted from Table 3, the gain from using the fractional model is noticeable since the fit to the observed yields, in terms of bias is (much) improved for the shorter maturities (1-2 years), while the model still fits the yields of bonds with longer maturities with deviation ranging from two to eight basis points. The average absolute bias for the fractional model is 4.72 basis points (bp) while it is 12.01 bp for the ordinary Vasicek model. This considerable improvement comes with only a minor increase in the average RMSE since the numbers are 45.20 bp and 41.55 bp, respectively.

9 Conclusion

In this paper, we have proposed a new way of dealing with the limitations of the classical affine dynamic term structure models (DTSMs). In particular, this paper has expanded the flexibility of the DTSMs by applying a fractional Brownian motion as the governing force of the state variable instead of the standard Brownian motion. This is a new direction in pricing non defaultable bonds with offspring in the arbitrage free pricing of weather derivatives based on fractional Brownian motions. By application of fractional Itô calculus and a fractional version of the Girsanov transform, we derived a no arbitrage price of the bond by solving a fractional version of the fundamental bond pricing equation. Besides this theoretical contribution, the paper proposed an estimation methodology based on the Kalman filter approach, which applied to weekly observations of US Treasury rates revealed that the instantaneous interest rate is fractionally integrated of order 0.05, $I(0.05)$, with relatively fast mean reversion, $\kappa = 0.22$. This combination allows a fairly precise prediction of all the bond yields used in the analysis, since the deviations to the observed yields are small for all maturities. In particular, the average absolute bias for the fractional model is 4.72 basis points while it is 12.01 basis points for the ordinary Vasicek model. This considerable improvement were obtained with only a minor increase in the average RMSE.

An advantage of our setup, not investigated here, but left for future research, is that the fractional version of the classical Vasicek model implies that the realized bond return or yield variation is long range dependent, which is a well known feature of interest rate series.

Hence our setup captures the properties that interst rates might be integrated of order d, see Shea (1991), Backus & Zin (1993), Crato & Rothman (1994), Høg (1997), Tkacz (2001), Iglesias & Phillips (2005), Nielsen (2006), and also that the yield variation is long range dependent.

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Table 1: Summary Statistics for the US Treasury Zero Coupon Yields, 1991 - 2002

Maturity	Mean	Std. dev.	Skewness	Kurtosis	p-value	$\tilde{d}$ ($m = [n^{0.5}]$)	$\tilde{d}$ ($m = [n^{0.7}]$)
1-year	4.90	1.04	-0.76	2.91	0.00	0.93	1.04
2-year	5.25	0.98	-0.54	2.98	0.00	0.90	1.08
4-year	5.71	0.88	-0.03	2.74	0.44	0.95	1.10
6-year	5.97	0.86	0.19	2.65	0.05	1.03	1.09
8-year	6.19	0.86	0.26	2.48	0.00	1.09	1.07
10-year	6.35	0.86	0.26	2.32	0.00	1.11	1.06

The table shows summary statistics for the US Treasury zero coupon yields. The p-values are from Jarque-Bera normality test and the long memory estimates are based on the Gaussian semiparametric estimator of Robinson (1995).

Table 2: The Empirical Results for the One Factor Models.

	The OUP	The FOUP
Parameter	Estimate	Estimate
d	$-$	0.0418
		(0.0324)
κ	0.0158	0.2189
	(0.0059)	(0.0159)
θ	0.0529	0.0416
	(0.2541)	(0.2110)
σ	0.0109	0.0125
	(0.0002)	(0.0002)
λ	-0.2700	-0.6034
	(0.0144)	(0.0392)

Note: The numbers in parenthesis are Whites heteroskedasticity consistent standard errors.

Table 3: The Accuracy Measures for the One Factor Vasicek Model.

	The OUP		The FOUP	
YTM	Bias (bp)	RMSE (bp)	Bias (bp)	RMSE (bp)
1	-41.60	95.24	-9.09	32.57
2	-21.02	64.93	-0.15	15.38
4	-1.81	26.38	3.79	37.76
6	-0.24	14.65	-2.00	53.29
8	-1.29	20.20	-5.04	63.36
10	-6.12	27.87	-8.26	68.84

Note: The table shows the two one factor models' fit in terms of bias and RMSE in basis points.