Linear-Quadratic Approximation, Efficiency and Target-Implementability*

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Abstract

We examine linear-quadratic (LQ) approximation of stochastic dynamic optimization problems in macroeconomics (and elsewhere), in particular in policy analysis using Dynamic Stochastic General Equilibrium (DSGE) models. We first define the problem that is solved by a social planner, given that the objective of the latter is to maximize average welfare; this yields the efficient solution. We then comment on the LQ approximation when a tax or subsidy can be imposed such that the zero-inflation competitive steady state output level is equal to the efficient level. We then examine the correct procedure for replacing a stochastic non-linear dynamic optimization problem with a linear-quadratic approximation. We show that a procedure proposed by Benigno and Woodford (2003) for large underlying distortions in the economy can be more easily implemented through a second-order approximation of the Hamiltonian used to compute the ex ante optimal policy with commitment (the Ramsey problem). We then define the notion of Target-Implementability, which is also a sufficient condition for a particular steady-state maximum of the Ramsey problem, and explain the usefulness of this in the context of stabilization policy.

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1 Introduction

Linear-quadratic (LQ) approximations to non-linear dynamic optimization problems in macroeconomics are widely used for a number of reasons. First, the characterization of time consistent and commitment equilibria for a single policy maker, and even more so for many interacting policymakers, are well-understood. Second, the certainty equivalence property results in optimal rules that are independent of the variance-covariance matrix of additive disturbances. Third, policy can be conveniently decomposed into deterministic and stochastic components. Fourth, the stability of the system is conveniently summarized in terms of eigenvalues. Finally for sufficiently simple models, linear-quadratic approximation allows analytical rather than numerical solution.

But what is the correct procedure for replacing a stochastic non-linear optimization problem with a linear-quadratic approximation? In Section 2 we begin by reviewing the setup of the benevolent policymaker's (or social planner's) problem whose solution yields what is termed the efficient output level. In addition we explain why the standard linear-quadratic approximation is appropriate for analysing optimal inflation policy in the decentralized economy (i.e., for the Ramsey problem), with the proviso that there is a tax (or subsidy) that ensures that the zero-inflation output level, or natural rate, exactly or approximately in some sense matches the efficient output level.

In Section 3 we turn to the general Ramsey problem, when there is no 'optimal' tax, that yields the 'distorted steady state' as it is termed by Benigno and Woodford (2004). We implement the LQ-approximation by quadratifying the Hamiltonian of the optimal control problem about the steady state. This idea stems from the sufficient conditions for the solution to an optimal control to be a maximum; Magill (1977) appears to be the first to have written up this result in the economics literature. We provide two simple examples of this, one of which relates to the procedure used by Benigno and Woodford (2003), henceforth BW, using a simple New Keynesian model and ad hoc policymaker's utility function as set out in Clarida *et al.* (1999). We demonstrate that for this simple example the BW procedure is equivalent to the Hamiltonian approach.

Section 4 then focuses on a third example: a Ramsey problem based on the New Keynesian framework of Section 2, with habit in consumption, with the policymaker adopting the utility function of consumers. We derive the corresponding LQ approximation to the policymaker's problem, and briefly comment on its representation when there is no habit. Section 5 defines the notion of Target-Implementability; this is essentially about the setting of targets by the Central Bank when it engages in stabilization. We show that this is equivalent to a requirement that the quadratic approximation be negative semi-definite, which is a sufficient but not necessary condition for optimality. We then obtain sufficient conditions for both target implementability and for the Ramsey problem to have a zero-inflation steady state and therefore natural rate of output.

2 The Social Planner's Problem and the Ramsey Problem assuming Efficiency

In this section we introduce the general form of the problem to be studied. We assume a set of consumers, each with given endowments, whose objective is to maximize an intertemporal utility function. Typically this will incorporate consumption and leisure, but we shall state the objectives in a general fashion, so that they can incorporate habit as well. Thus the objective is for individual i to maximize an expected utility function of the form

$$E_0 \sum_{t=0}^{\infty} \beta^t u(W_{it}; W_{t-1}) \tag{1}$$

where the vector W_{it} represents individual i's choices e.g. consumption and labour supply. This utility function may also incorporate habit or catching-up, and may therefore also be dependent on aggregate or average choices made in the previous period W_{t-1}. There are various resource constraints that we shall come to later.

Typically in economic models of this type we would assume monopolistic competition by firms, which leads to mark-up pricing, and creates a wedge between the level of output under competition - the natural rate - and the level of output that could be achieved by a social planner - the efficient level. This wedge may be exacerbated if we assume that there is labour market power as well. The latter is not incorporated by Benigno and Woodford (2004), but is common in most other New Keynesian models e.g. Clarida *et al.* (2002). We also assume that costs for firms are continuous, which rules out state-dependent $S-s$ policies; we do this because such policies cannot be easily aggregated. Initially we ignore the stochastic problem because the deterministic problem is sufficient to set up the LQ approximation.

Finally we assume that the resource constraints sum to an aggregate resource constraint. One can then define the social planner's problem in terms of the representative individual as that of

$$max \sum_{t=0}^{\infty} \beta^t U(X_{t-1}, W_t) \ \ s.t. \ \ X_t = f(X_{t-1}, W_t) \tag{2}$$

where the set of constraints in this problem represent the set of (intertemporal) resource and other relevant constraints. Although there appear to be significant differences in the functions u of (1) and U of (2), these are merely cosmetic; W_t would appear in the same way in U as did W_{it} in u, and W_{t-1} is now a subset of the X_t variables. Thus if we represent the vector $X_t = [X_{1t}^T \ X_{2t}^T]$, where X_1 represent the resource constraints, then $X_{2t} = W_t$, and $X_{2,t-1}$ appears in the same way in U as did W_{t-1} in u.

2.1 Characterization of the Efficient Level

Defining the Lagrangian

$$\sum_{t=0}^{\infty} \beta^t [U(X_{t-1}, W_t) - \lambda_t^T (X_t - f(X_{t-1}, W_t))] \tag{3}$$

the following first order conditions provide the necessary conditions for the solution:

$$U_W(X_{t-1}, W_t) + \lambda_t^T f_W(X_{t-1}, W_t) = 0 \qquad U_X(X_{t-1}, W_t) + \lambda_t^T f_X(X_{t-1}, W_{t+1}) - \frac{1}{\beta}\lambda_{t-1}^T = 0 \tag{4}$$

The steady state of the social planner's problem, the efficient level (denoted by $*$), is then given by

$$X^* = f(X^*, W^*) \qquad U_W(X^*, W^*) + \lambda^{*T} f_W(X^*, W^*) = 0$$
$$U_X(X^*, W^*) + \lambda^{*T} f_X(X^*, W^*) - \tfrac{1}{\beta}\lambda^{*T} = 0 \tag{5}$$

2.2 The Flexible-Price Solution and the Ramsey Problem

The difference between the efficient solution and that of the competitive or flexible-price solution is due to the externalities of habit and of firm and labour market power. As we shall see below for a particular example, the externality due to consumption habit works in the opposite direction to the externalities that produce the mark-ups in prices and wages.

In principle it is possible to set a proportional tax (or subsidy) in the flexible-price case that yields a 'natural' level of output exactly equal to the efficient level of output of the social planner.

Thus far we have only discussed the efficient and flexible-price levels of output. A more general model takes into account the fact that neither wages nor prices are completely flexible. As a consequence, we must discuss the case where a policymaker is required to maximize average welfare, in this case by choosing the optimal path for inflation. This is a particular case of the Ramsey problem.

The standard New Keynesian model ascribes a fixed probability in each period of changing prices (and wages). This leads to dynamic equations for the overall price index, and in turn this leads in the Woodford (2003) case to different choices of labour supply by individuals, and in the Clarida *et al.* (2002) case to each individual providing the same quantity of labour. In the former, the policymaker takes the average of the utility function, which for small variance of shocks is approximately the same as flexible-price level of the utility function, but with an additional effect from the spread of prices. In the latter, although labour supply is the same for each worker, it is dependent on the spread of demand for each good; this in turn leads to the utility function differing from the flexible price utility function by a term dependent on the spread of prices and wages. From the point of view of the Ramsey policymaker, the problem can then be written approximately as maximizing

$$\sum_{t=0}^{\infty} \beta^t [U(X_{t-1}, W_t) + A(X_{t-1}, W_t) D_t^P] \tag{6}$$

where D_t^P represents the spread of prices, and we ignore the spread of wages to ease the algebraic burden. In the case of standard New Keynesian models, the term $A(X, W)$ contains the disutility of labour because that is what is affected by price variability. This Ramsey problem is subject to the resource constraints above, and in addition to further constraints representing individual consumer and firm behaviour (arising for example from staggered price and wage-setting). We assume that these can be aggregated, so that the constraints that must be satisfied by the Ramsey policymaker constitute both the resource constraints and constraints associated with price-setting:

$$X_t = f(X_{t-1}, W_t) \qquad Z_t = g(Z_{t-1}, X_{t-1}, W_t; \tau) \tag{7}$$

Woodford (2003), among others, shows that the spread of prices D_t^P is approximately obtained as a quadratic form in current and past levels of inflation, where inflation is one of the variables included in the vector Z_t.

It is important to appreciate that the the constraints associated with Z_t represent individuals' and firms' decisions, and may involve future expectations. We take the approach that the policymaker here has reputation for precommitment, so that we can take expectations of the future as always being fulfilled, and therefore regard these equations as backward looking. Secondly, if all factor prices are fixed so that inflation is 0 i.e. the appropriate elements of the vector Z are set equal to 0, we obtain a solution to the 'natural' rate by solving for the steady state $\bar{X} = f(\bar{X}, \bar{W})$, $\bar{Z} = g(\bar{Z}, \bar{X}, \bar{W}; \tau)$. This is also known as the flexible price equilibrium. An important consideration is that the natural rate will be dependent on the tax/subsidy rate τ.

2.3 LQ Approximation of the Ramsey Problem: Efficient Case

Woodford (2003) now points out a key result for LQ-approximation. If at all possible, the aim of the Ramsey policymaker is to stabilize the economy about the efficient level of output. Let us assume therefore that the proportional tax/subsidy is set at exactly the level at which the flexible price equilibrium achieves the efficient level of output. This implies that there exists a value τ^* such that the efficient rate, coupled with zero inflation, is a solution to $Z^* = g(X^*, Z^*, W^*; \tau^*)$.

The main result of this section is dependent on the ability (a) to expand the utility function about the steady steady state efficient solution without the presence of linear terms and (b) to expand the constraints about the steady state efficient solution without the presence of constant terms.

Theorem 1: The stabilization problem for the Ramsey policymaker can be approximately expressed as a quadratic expansion of the welfare function about the efficient level.

Proof: We first deal with the utility function:

$$
\begin{aligned}
& \sum_{t=0}^{\infty} \beta^t [U(X_{t-1}, W_t) + A(X_{t-1}, W_t) D_t^P] \\
= & \sum_{t=0}^{\infty} \beta^t [U(X_{t-1}, W_t) - \lambda^{*T}(X_t - f(X_{t-1}, W_t)) + A(X_{t-1}, W_t) D_t^P] \\
\cong & \sum_{t=0}^{\infty} \beta^t [U(X^*, W^*) + U_X \delta X_{t-1} + U_W \delta W_t - \lambda^{*T}(\delta X_t - f_X \delta X_{t-1} - f_W \delta W_t) \\
& + \frac{1}{2}(\delta X_{t-1}^T H_{XX} \delta X_{t-1} + 2\delta X_{t-1}^T H_{XW} \delta W_t + \delta W_t^T H_{WW} \delta W_t + A(X^*, W^*) D_t^P] \\
= & \sum_{t=0}^{\infty} \beta^t [U(X^*, W^*) + (U_X - \frac{1}{\beta}\lambda^{*T} + \lambda^{*T} f_X)\delta X_{t-1} + (U_W + \lambda^{*T} f_W)\delta W_t \\
& + \frac{1}{2}(\delta X_{t-1}^T H_{XX} \delta X_{t-1} + 2\delta X_{t-1}^T H_{XW} \delta W_t + \delta W_t^T H_{WW} \delta W_t + A(X^*, W^*) D_t^P] \quad (8)
\end{aligned}
$$

where $H = U(X, W) + \lambda^{*T} f(X, W)$, and its second derivatives are evaluated at(X^*, W^*). Hence, using (5), the linear terms in $\delta X_t, \delta W_t$ vanish. We shall see below a representation of D_t^P that allows us to write the contribution from inflation as a simple quadratic term in the utility function for each period t; this is why we were able to ignore first order changes in the function $A(X, W)$.

Now consider the constraints. Firstly the resource constraint is in steady state at the efficient level, so that an expansion about the latter will contain no constant term. Secondly, the constraint involving Z, by appropriate choice of τ is also in a zero-inflation steady state at the efficient level, so that any approximation of its dynamics about the efficient level will be without a constant term.

The implication of this proof is that the welfare function can always be approximated as a constant plus quadratic terms, centred on the efficient rate, once the resource constraints have been incorporated. It is only the equations describing private sector behaviour that can make invalid this LQ approximation to stabilization.

2.4 The Small Distortion Case

Suppose that the tax/subsidy is insufficient to eliminate the inefficiency, but that the latter is small. There are then two approaches to obtain an approximation to the LQ approximation. The first is take deviations about the inefficient steady state. This will, as we have seen above, produce an approximation to the welfare that contains a constant term (the welfare in the efficient case), and a quadratic term. The error in the approximation is then in the dynamic equation describing individual decisions. This is because we need a vector of constants to be included in the dynamic equation for deviations in Z_t, which is given by $\bar{Z} - g(\bar{X}, \bar{Z}, \bar{W}; \tau)$;if this is small, it may be ignored.

The alternative is to take deviations about the natural rate, as done by Woodford (2003), Appendix E. The dynamic equations in deviation form then no longer contain a constant, but the linear terms in the welfare approximation (8) are now of the form:

$$\begin{aligned}
& (U_X(\bar{X}, \bar{W}) - \frac{1}{\beta}\lambda^{*T} + \lambda^{*T} f_X(\bar{X}, \bar{W}))\delta X_{t-1} + (U_W(\bar{X}, \bar{W}) + \lambda^{*T} f_W(\bar{X}, \bar{W}))\delta W_t \\
= \; & (H_X + (\bar{X} - X^*)^T H_{XX} + (\bar{W} - W^*)^T H_{WX})\delta X_{t-1} \\
& + (H_W + (\bar{X} - X^*)^T H_{XW} + (\bar{W} - W^*)^T H_{WW})\delta W_t \\
= \; & ((\bar{X} - X^*)^T H_{XX} + (\bar{W} - W^*)^T H_{WX})\delta X_{t-1} \\
& + ((\bar{X} - X^*)^T H_{XW} + (\bar{W} - W^*)^T H_{WW})\delta W_t
\end{aligned} \tag{9}$$

Thus the linear terms can be ignored provided that $\bar{X} - X^*$ and $\bar{W} - W^*$ are small.

We assess the limitations of the small distortion case in Section 4 by comparing the weights on the quadratic terms of the LQ welfare approximation for the efficient and the non-efficient case. This provides an arguably more direct assessment of the error in the approximation; this is because it is less easy to assess the impact of the errors described above.

3 The Hamiltonian LQ Approximation for Large Distortions

In general, one cannot expect fiscal authorities to set a tax/subsidy so as to achieve the efficient level of output. This means that the LQ Approximation to the utility function of the previous section will be inappropriate. A general statement of a Ramsey problem in economics involves both backward-looking dynamics such as capital accumulation, and forward-looking dynamics such as the Euler equation for consumption or, as below, an equation for aggregate inflation in which the latter depends on expectations of future inflation. Benigno and Woodford (2003) solve the stabilization aspect of such a problem by expanding about the precommitment deterministic solution. They invoke a rather tortuous method that is not obviously generalised. However there does exist a generalisation due in part to Magill (1977). As a preliminary, instead of writing down a general economic model that incorporates both backward and forward looking behaviour, we note that our intention is to obtain an LQ approximation to the precommitment solution. Since formally this solution makes no distinction between a variable dated at $t+1$ and its expectation using information at time t, we can for the moment write down the general model as purely backward-looking.

We now adopt a slight change of notation. Since in this section we are no longer interested in approximating about the efficient level, the resource constraints do not play the special role that they did in the section above. We therefore absorb all state variables X, Z into just one state vector X, so that the general policymaker's problem no longer requires a special term representing the spread of prices.

The general dynamic programming problem is therefore:

$$Max \sum_{t=0}^{\infty} \beta^t U(X_{t-1}, W_t) \;\; s.t. \;\; X_t = f(X_{t-1}, W_t) \tag{10}$$

Define the Hamiltonian $H_t = U(X_{t-1}, W_t) + \lambda_t^T f(X_{t-1}, W_t)$, where λ_t are the Lagrange multipliers for the constraints, as in (3). The following is the discrete time version of Magill (1977):

Theorem 2:

(a) If a steady state solution $(\bar{X}, \bar{W}, \bar{\lambda})$ to (10) exists, then any perturbation $(\delta X_t, \delta W_t)$ about this steady state can be expressed as the solution to

$$Max \sum_{t=0}^{\infty} \beta^t \begin{bmatrix} \delta X_{t-1} & \delta W_t \end{bmatrix} \begin{bmatrix} H_{XX} & H_{XW} \\ H_{WX} & H_{WW} \end{bmatrix} \begin{bmatrix} \delta X_{t-1} \\ \delta W_t \end{bmatrix} \;\; s.t. \;\; \delta X_t = f_X \delta X_{t-1} + f_W \delta W_t \tag{11}$$

where all derivatives are evaluated at $(\bar{X}, \bar{W})$.

(b) A necessary and sufficient condition for the dynamic programming problem (10) to constitute a maximum with steady state $(\bar{X}, \bar{W})$ is that the steady state Riccati matrix associated with (11) is negative definite.

Note that the perturbed system is now in standard linear-quadratic format, which is the basis for (b). Of course, if the function $f(X,W)$ is linear and $U(X,W)$ is concave then (b) is irrelevant. However it is unlikely that f will be linear, so we have the following, possibly stringent, condition:

Result 1: A sufficient condition for for the steady state to be a maximum is that the matrix of second derivatives of H in (11) is negative semi-definite[1].
Magill (1977)'s result extends to the stochastic case as well. Thus if the dynamic equations are written as $X_t = f(X_{t-1}, W_t, \varepsilon_t)$, where the ε_t have mean zero and are independently normally distributed then any perturbations about the deterministic solution are solutions to the problem

$$Max\ E_0 \quad \sum_{t=0}^{\infty} \beta^t \begin{bmatrix} \delta X_{t-1} & \delta W_t & \varepsilon_t \end{bmatrix} \begin{bmatrix} H_{XX} & H_{XW} & H_{X\varepsilon} \\ H_{WX} & H_{WW} & H_{X\varepsilon} \\ H_{\varepsilon X} & H_{\varepsilon W} & H_{\varepsilon\varepsilon} \end{bmatrix} \begin{bmatrix} \delta X_{t-1} \\ \delta W_t \\ \varepsilon_t \end{bmatrix}$$
$$s.t. \quad \delta X_t = f_X \delta X_{t-1} + f_W \delta W_t + f_\varepsilon \varepsilon_t \tag{12}$$

Before turning to the main model of the paper, we provide two examples of the Hamiltonian approach.

Example 1: We obtain a second-order accurate solution to the following problem that motivates Kim and Kim (2003):

$$Max\ lnC_1 + lnC_2 \quad s.t.\ C_1 + C_2 = Y_1 + Y_2 \tag{13}$$

where $lnY_i \sim N(0, \sigma^2)$ The solution to this is clearly $C_i = (Y_1 + Y_2)/2$, so that the deterministic solution centred at $Y_1 = Y_2 = 1$ is $C_1 = C_2 = 1$. If we define the Lagrangian as $L = lnC_1 + lnC_2 - \lambda(C_1 + C_2 - Y_1 - Y_2)$, then the value of λ at the optimum is clearly 1. Using perturbations of the logs of C_i, Y_i, it is easy to show that the problem transforms to

$$Max\ c_1 + c_2 + \frac{1}{2}(y_1 + y_2 + y_1^2 + y_2^2 - c_1 - c_2 - c_1^2 - c_2^2) \quad s.t.\ c_1 + c_2 = y_1 + y_2 \tag{14}$$

[1]A simple example of a problem for which a maximum exists, but for which the sufficient condition does not hold is: max $x^2 - y^2$ such that $y = ax + b$. It is easy to see that the stationary point is a maximum when $|a| > 1$.

which has solution $c_i = (y_1 + y_2)/2$, so that the maximand is equal to $y_1 + y_2 + \frac{1}{4}(y_1 - y_2)^2$, as in Kim and Kim (2003).

Example 2: In this example we briefly outline the Benigno and Woodford (2004) approach for a monetary policy problem, and outline its equivalence to the Hamiltonian method.

Consider the following optimization problem for a monetary authority to choose state-contingent path for its inflation target π_t so as to minimize an ad hoc objective function

$$E_0 \left[\sum_{t=0}^{\infty} \lambda^t \left[(x_t - x^*)^2 + \pi_t^2 \right] \right] \tag{15}$$

where x_t is the output gap in logarithms given by a 'New Keynesian' Phillips curve

$$\pi_t = \beta E_t \pi_{t+1} + f(x_t) + u_t \,; \;\; f' > 0, \; f'' < 0 \tag{16}$$

where u_t is an i.i.d. supply shock. Let x_n be the natural rate of output defined by $f(x_n) = 0$. Then (15) $x^* \geq x_n$ is the logarithm of the efficient level of output where inefficiency arises from monopolistic competition in the output market.

A common procedure for reducing this to a LQ problem is to expand about the steady state x so that $f(x_t) \simeq f(x) + f'(x)(x_t - x) + \frac{1}{2} f''(x)(x_t - x)^2 {=} a(x_t - x) - b(x_t - x)^2$. Much of the literature[2] including Clarida *et al.* (1999) then erroneously adopts a linearized Phillips curve

$$\pi_t = \beta E_t \pi_{t+1} + a(x_t - x) + u_t \tag{17}$$

and proceeds with the LQ problem of minimizing (15) subject to (17). The error arises from the objective function including a linear term in $x_t x^*$. From (16), $x_t = f^{-1}(\pi_t - \beta E_t \pi_{t+1} - u_t)$ so unless x^* is small, there is a second-order term missing in the objective function if one proceeds with the linear approximation (17). To get round this problem, the procedure set out in BW selects a new steady state $(\bar{\pi}, \bar{x})$ satisfying (16) and a multiplier h such that

$$\sum_{t=0}^{\infty} \lambda^t [\theta(x_t - \bar{x})^2 + \phi(\pi_t - \bar{\pi})^2] \equiv \sum_{t=0}^{\infty} \lambda^t [(x_t - x^*)^2 + \pi_t^2 + h[\beta(\pi_{t+1} - \bar{\pi}) - (\pi_t - \bar{\pi}) - f(x_t) + f(\bar{x})]] \tag{18}$$

[2]Some previous work of one of the authors joins a distinguished list (see Currie and Levine (1993).

up to a second order approximation in deviations about the steady state, give or take constant terms. Then the problem becomes that of minimizing (18) subject to

$$\pi_t - \bar{\pi} = \beta E_t \pi_{t+1} - \bar{\pi}) - f(x_t) + f(\bar{x}) + u_t \cong \beta E_t \pi_{t+1} - \bar{\pi}) + a(x_t - \bar{x}) + u_t \tag{19}$$

The BW procedure then amounts to finding the values $\bar{\pi}, \bar{x}, \theta, \phi$ and h which are consistent with the equalities in (18) and (19).

Using the Hamiltonian approach it is easy, but less tedious, to show that $\bar{\pi}, \bar{x}$ and h are given by

$$2(\bar{x} - x^*) - hf'(\bar{x}) = 0 \qquad 2\bar{\pi} + h\left(1 - \frac{\beta}{\lambda}\right) = 0 \qquad (1-\beta)\bar{\pi} - f(\bar{x}) = 0 \tag{20}$$

Details are provided in Appendix A.

4 Linear-Quadratic Approximation of Welfare in a DSGE Model

The model is the cashless economy as in Batini *et al.* (2006) with habit in consumption. Agents (or consumers) of type i maximize the intemporal trade-off between consumption C_{it} - taking into account a desire to consume at a level similar to that of last period's average consumption C_{t-1} - and leisure. The latter is accounted for by penalising working time N_{it}.

Unlike Clarida *et al.* (2000) we do not incorporate a proportional tax (or subsidy) into the model in order to ensure that the steady state, or natural rate, of output is at the efficient level[3]. Instead we use the methodology of the previous section to show how to obtain a quadratic approximation to the welfare when the natural rate differs from the efficient rate. This is an issue also addressed by Benigno and Woodford (2004) using the the methods of Section 2.1.

We can summarize the model in a concise form as:

Household Utility

$$\Omega_0 = E_0 \left[\sum_{t=0}^{\infty} \beta^t \left[\frac{(C_{it} - h_C C_{t-1})^{1-\sigma}}{1-\sigma} - \kappa \frac{N_{it}^{1+\phi}}{1+\phi} \right] \right] \tag{21}$$

[3]see Section 4.1 below

Household Behaviour The first-order conditions for households are as follows:

$$1 = \beta E_t \left[Q_{t,t+1} \left(\frac{C_{it+1} - h_C C_t}{C_{it} - h_C C_{t-1}} \right)^{-\sigma} \frac{1}{\Pi_{t+1}} \right] \tag{22}$$

$$\frac{W_{it}}{P_t} = \frac{\kappa}{(1 - \frac{1}{\eta})} N_{it}^{\phi} (C_{it} - h_C C_{t-1})^{\sigma} \tag{23}$$

where $Q_{t,t+1}$ is the expected value of the stochastic discount factor on holdings of one-period bonds, and the gross inflation rate Π_t is given by

$$\Pi_t \equiv \frac{P_t}{P_{t-1}} \tag{24}$$

All consumers can trade in a complete set of state contingent bonds, and therefore engage in complete risk-sharing, so that (22) represents the Keynes-Ramsey intertemporal first-order condition for consumption across all consumers, taking habit into account. (23) equates relative marginal utilities of consumption and leisure to the real wage. W_{it}, P_t are measures of the nominal wage of the ith agent and of price respectively. (23) also incorporates market power of individual consumers, who are all distinct from the point of view of production skills, so that the elasticity of substitution between them corresponds to an elasticity of demand for their services denoted by η. Underlying this is an assumption that output is CES in labour, with its level expressed by aggregating over all labour inputs.

We make the simplest possible assumption here that there are no lags in wage-setting; as a consequence there is market-clearing in wages, with all agents setting the same wage and all working for the same number of hours. Thus (23) holds when i is deleted, so for this setup there is no need to aggregate W_t, N_t via the elasticity of demand for labour η.

Firms:

Unlike workers, firms only reset prices in any given period with probability $1 - \xi$. Thus the optimal price P_t^0 for any firm that sets its price at t must take into account any future periods during which the price remains unchanged.[4]

The first-order condition for profit-maximization for the jth firm over the duration of the optimal price not being reset takes into account the elasticity of substitution ζ between

[4] It is easy to show that if there is planned indexation to the overall price index as well i.e. the future price at time $t + k$ is given by $P_t^0 (P_{t+k-1}/P_t)^{\gamma}$ then all the results presented here are the same when Π_t is replaced by $\Pi_t / \Pi_{t-1}^{\gamma}$.

goods, which provides firms with monopolistic power. It is given by

$$P_t^0 E_t[\sum_{k=0}^{\infty} \xi^k Q_{t,t+k} Y_{t+k}(j)] = \frac{\kappa}{(1-1/\zeta)} E_t[\sum_{k=0}^{\infty} \xi^k Q_{t,t+k} P_{t+k} MC_{t+k} Y_{t+k}(j)]$$

where marginal cost is given by the real product wage $MC_t = \frac{W_t}{A_t P_t}$ and the stochastic discount factor $Q_{t,t+k}$ is given by

$$Q_{t,t+k} = \beta^k (\frac{C_{t+k} - h_C C_{t+k-1}}{C_t - h_C C_{t-1}})^{-\sigma} \frac{P_t}{P_{t+k}} \tag{25}$$

Noting that

$$Y_{t+k}(j) = \left(\frac{P_t^0}{P_{t+k}} \right)^{-\zeta} Y_{t+k} \tag{26}$$

and multiplying both sides of (25) by $(\frac{P_t^0}{P_t})^{\zeta}(C_t - h_C C_{t-1})^{-\sigma}$ and in addition noting that $P_{t+k}/P_t = \Pi_{t+k}...\Pi_{t+1}$, then it is straightforward to express the solution to this problem as follows:

Define variables Q_t, H_t and Λ_t by

$$\begin{aligned} Q_t &\equiv P_t^0/P_t & (27) \\ H_t - \xi\beta E_t[\Pi_{t+1}^{\zeta-1} H_{t+1}] &= Y_t(C_t - h_C C_{t-1})^{-\sigma} & (28) \\ \Lambda_t - \xi\beta E_t[\Pi_{t+1}^{\zeta} \Lambda_{t+1}] &= \frac{\kappa}{(1-1/\zeta)(1-1/\eta)A_t} Y_t N_t^{\phi} & (29) \end{aligned}$$

Then the firms' staggered price setting can be succinctly described by

$$Q_t = \Lambda_t / H_t \tag{30}$$

with price index inflation given by

$$1 = \xi \Pi_t^{\zeta-1} + (1-\xi) Q_t^{1-\zeta} \tag{31}$$

Note that we have not yet determined the relationship between total output and the aggregate measure of labour input. However at the firm level we can define it for firm k as

$$Y_t(k) = A_t N_t(k) \tag{32}$$

where A_t represents a common technology shock.

4.1 Effects of Inflation

Here we discuss the effects of inflation on the dispersion of prices due to firms' behaviour discussed above, and the implications for total employment. These dispersion effects will lead to costs of inflation, as we shall see later.

Woodford (2003) has demonstrated the effect on price dispersion of inflation, and derived the following relationship for the variance of the log of prices:

$$D_t = \xi D_{t-1} + \frac{\xi}{1-\xi}(ln\Pi_t)^2 \tag{33}$$

The impact of price dispersion arises from labour input being the same for each individual, but dependent on demand for each good:

$$N_t = \sum N_t(j) = \frac{Y_t}{A_t}\sum \frac{Y_t(j)}{Y_t} = \frac{Y_t}{A_t}\sum \left(\frac{P_t(j)}{P_t}\right)^{-\zeta} \tag{34}$$

Now assume that that $lnP_t(j)$ is approximately normally distributed as $N(\mu_t, D_t)$, which is a relatively innocuous assumption for ξ close to 1; by the law of large numbers, it follows that the overall price index P_t is given by

$$P_t^{1-\zeta} = \sum P_t(j)^{1-\zeta} = E[e^{(1-\zeta)lnP_t(j)}] = e^{(1-\zeta)\mu_t + \frac{1}{2}(1-\zeta)^2 D_t} \tag{35}$$

Similarly one can obtain an approximate expression for the last term of (34):

$$\sum \left(\frac{P_t(j)}{P_t}\right)^{-\zeta} = \sum P_t^{\zeta} E[e^{(-\zeta lnP_{it}}] = e^{\zeta\mu_t + \frac{1}{2}\zeta(1-\zeta)D_t} e^{-\zeta\mu_t + \frac{1}{2}\zeta^2 D_t} = e^{\frac{1}{2}\zeta D_t} \tag{36}$$

From this it follows that

$$N_t^{\phi} \cong \frac{Y_t^{\phi}}{A_t^{\phi}} e^{\frac{1}{2}\phi\zeta D_t} \cong \frac{Y_t^{\phi}}{A_t^{\phi}}(1 + \frac{1}{2}\phi\zeta D_t) \tag{37}$$

4.2 The Ex Ante Optimal (Ramsey) Problem

As a consequence of the price diversion result above, the problem for a policy maker is characterised by solving the *deterministic* ex ante (commitment and Ramsey) problem by choosing a trajectory for inflation to maximize

$$\Omega_0 = \sum_{t=0}^{\infty} \beta^t \left[\frac{(Y_t - Z_t)^{1-\sigma}}{1-\sigma} - \frac{\kappa}{1+\phi}\left(\frac{Y_t}{A_t}\right)^{1+\phi}\left(1 + \frac{1}{2}\zeta(1+\phi)D_t\right)\right] \tag{38}$$

subject to the constraints

$$Z_t = h_C C_{t-1} \qquad 1 = \xi\Pi_t^{\zeta-1} + (1-\xi)Q_t^{1-\zeta} \qquad Q_t H_t = \Lambda_t \tag{39}$$

$$H_t - \xi\beta E_t[\Pi_{t+1}^{\zeta-1} H_{t+1}] = Y_t(Y_t - Z_t)^{-\sigma} \tag{40}$$

$$\Lambda_t - \xi\beta E_t[\Pi_{t+1}^{\zeta} \Lambda_{t+1}] = \frac{\kappa}{(1-1/\zeta)(1-1/\eta)} \left(\frac{Y_t}{A_t}\right)^{1+\phi} (1 + \frac{1}{2}\zeta\phi D_t) \tag{41}$$

$$D_t = \xi D_{t-1} + \frac{\xi}{1-\xi}(ln\Pi_t)^2 \tag{42}$$

We can now write the Lagrangian for the policymaker's optimal control problem as follows:

$$\begin{aligned} L &= \Omega_0 + \sum_{t=0}^{\infty} \beta^t [\lambda_{1t}(Z_t - h_C Y_t) + \lambda_{2t}(1 - \xi\Pi_t^{\zeta-1} - (1-\xi)Q_t^{1-\zeta}) \\ &+ \lambda_{3t}(Q_t H_t - \Lambda_t) + \lambda_{4t}(H_t - \xi\beta\Pi_{t+1}^{\zeta-1} H_{t+1} - Y_t(Y_t - Z_t)^{-\sigma}) \\ &+ \lambda_{5t}(\Lambda_t - \xi\beta\Pi_{t+1}^{\zeta}\Lambda_{t+1} - \frac{\kappa}{\alpha}\left(\frac{Y_t}{A_t}\right)^{1+\phi}\left(1 + \frac{1}{2}\zeta\phi D_t\right)) \\ &+ \lambda_{6t}(D_t - \xi D_{t-1} - \frac{\xi}{1-\xi}(ln\Pi_t)^2)] \end{aligned} \tag{43}$$

where we define $\alpha = (1 - 1/\zeta)(1 - 1/\eta)$.

First-order conditions are given by:

$$\begin{gathered} (Y_t - Z_t)^{-\sigma} - \kappa\frac{Y_t^{\phi}}{A_t^{1+\phi}}(1 + \tfrac{1}{2}\zeta(1+\phi)D_t) - \lambda_{1t}h_C \\ -\lambda_{5t}\frac{\kappa(1+\phi)}{\alpha}\frac{Y_t^{\phi}}{A_t^{1+\phi}}(1 + \tfrac{1}{2}\zeta\phi D_t) - \lambda_{4t}((Y_t - Z_t)^{-\sigma} - \sigma Y_t(Y_t - Z_t)^{-\sigma-1}) = 0 \end{gathered} \tag{44}$$

$$-(Y_t - Z_t)^{-\sigma} + \frac{1}{\beta}\lambda_{1t-1} - \lambda_{4t}\sigma Y_t(Y_t - Z_t)^{-\sigma-1} = 0 \tag{45}$$

$$\beta(1-\zeta)\xi\lambda_{2,t+1}\Pi_{t+1}^{\zeta-2} - \lambda_{4t}\xi\beta(\zeta-1)\Pi_{t+1}^{\zeta-2}H_{t+1} - \lambda_{5t}\xi\beta\zeta\Pi_{t+1}^{\zeta-1}\Lambda_{t+1} - \frac{2\xi\beta}{1-\xi}\lambda_{6,t+1}\frac{ln\Pi_{t+1}}{\Pi_{t+1}} = 0 \tag{46}$$

$$-\lambda_{2t}(1-\xi)(1-\zeta)Q_t^{-\zeta} + \lambda_{3t}H_t = 0 \tag{47}$$

$$\lambda_{3t}Q_t + \lambda_{4t} - \xi\Pi^{\zeta-1}\lambda_{4,t-1} = 0 \tag{48}$$

$$-\lambda_{3t} + \lambda_{5t} - \xi\Pi_t^{\zeta}\lambda_{5,t-1} = 0 \tag{49}$$

$$-\frac{1}{2}\kappa\zeta\left(\frac{Y_t}{A_t}\right)^{1+\phi} + \lambda_{6t} - \xi\beta\lambda_{6,t+1} - \frac{\kappa\phi\zeta}{2\alpha}\left(\frac{Y_t}{A_t}\right)^{1+\phi}\lambda_{5t} = 0 \tag{50}$$

The zero-inflation equilibrium values are given by

$$\Pi = Q = 1 \qquad \Lambda = H = \frac{Y^{1-\sigma}(1-h_C)^{-\sigma}}{1-\beta\xi} \qquad D = 0 \qquad (1-h_C)^{-\sigma} = \frac{\kappa}{\alpha}\frac{Y^{\phi+\sigma}}{A^{1+\phi}} \tag{51}$$

$$\lambda_5 = \frac{1-\beta h_C - \alpha}{\frac{\sigma(1-h_C\beta)}{1-h_C}+\phi} = -\lambda_4 \qquad \lambda_3 = (1-\xi)\lambda_5 \qquad \lambda_2 = \frac{H\lambda_5}{(1-\zeta)} \tag{52}$$

$$\lambda_6 = \frac{\zeta Y^{1-\sigma}(1-h_C)^{-\sigma}}{2(1-\beta\xi)} \frac{\frac{\alpha\sigma}{1-h_C}+\phi}{\frac{\sigma(1-h_C\beta)}{1-h_C}+\phi} \tag{53}$$

Now that we have the steady-state values of the Lagrange multipliers, we are in a position to apply Theorem 2(a). We first linearize the relationships between the variables, and then obtain the quadratic approximation of the Lagrangian. We shall leave discussion of Theorem 2(b) till later.

4.3 Linearization of Dynamics

We linearize about a zero-inflation steady state. Define $h_t, \lambda_t, q_t, \pi_t$ as deviations of $H_t, \Lambda_t, Q_t, \Pi_t$ from their steady state values. In addition define $y_t = (Y_t - Y)/Y$, $a_t = (A_t - A)/A$ and define $z_t = (Z_t - Z)/Y$.

Linearization of the constraints yields

$$Hq_t = \lambda_t - h_t \qquad \xi\pi_t = (1-\xi)q_t \tag{54}$$

$$z_{t+1} = h_C y_t \tag{55}$$

$$h_t - \beta\xi(\zeta-1)HE_t\pi_{t+1} - \beta\xi E_t h_{t+1} = Y^{1-\sigma}(1-h_C)^{-\sigma}(y_t - \frac{\sigma}{1-h_C}(y_t - z_t)) \tag{56}$$

$$\lambda_t - \beta\xi\zeta\Lambda E_t\pi_{t+1} - \beta\xi E_t\lambda_{t+1} = \frac{\kappa(1+\phi)}{\alpha}\frac{Y^{1+\phi}}{A^{1+\phi}}(y_t - a_t) \tag{57}$$

Now subtract (56) from (57). Noting that $\Lambda = H$, and substituting from (54) yields a Phillips curve relationship of the form:

$$\pi_t = \beta E_t\pi_{t+1} + \frac{(1-\xi)(1-\beta\xi)}{\xi}(\phi y_t + \frac{\sigma}{1-h_C}(y_t - z_t) - (1+\phi)a_t) \tag{58}$$

Note that linearization of the dispersion term around zero inflation is irrelevant, since it reduces to $d_t - \xi d_{t-1} = 0$.

4.4 The Commitment Solution: Quadratification of Lagrangian

At this point we apply the result of Section 2, in order to obtain a quadratic approximation to the period t value of the Lagrangian. Ignoring the steady state value of the latter, the remaining terms are given by:

$$
\begin{aligned}
&-\frac{1}{2}(Y-Z)^{-\sigma-1}Y^2\sigma(y_t-z_t)^2-\frac{1}{2}\kappa\phi\frac{Y^{1+\phi}}{A^{1+\phi}}y_t^2-\lambda_5\frac{\kappa}{2\alpha}\phi(1+\phi)\frac{Y^{1+\phi}}{A^{1+\phi}}y_t^2\\
&+\kappa(1+\phi)\frac{Y^{1+\phi}}{A^{1+\phi}}y_ta_t+\lambda_5\frac{\kappa}{\alpha}(1+\phi)^2\frac{Y^{1+\phi}}{A^{1+\phi}}y_ta_t\\
&-\lambda_5\sigma Y^2(Y-Z)^{-\sigma-1}(y_t-z_t)y_t+\frac{1}{2}\lambda_5\sigma(\sigma+1)Y^3(Y-Z)^{-\sigma-2}(y_t-z_t)^2\\
&-\frac{\xi}{2}\pi_t^2((\zeta-1)(\zeta-2)\lambda_2\Pi^{\zeta-3}+(\zeta-1)(\zeta-2)\Pi^{\zeta-3}H\lambda_4+\zeta\lambda_5(\zeta-1)\Pi^{\zeta-2}\Lambda+\frac{2\lambda_6}{(1-\xi)\Pi^2})\\
&-\xi\pi_t\lambda_t\zeta\lambda_5\Pi^{\zeta-1}-\xi\pi_th_t(\zeta-1)\lambda_4\Pi^{\zeta-2}\\
&+\frac{1}{2}q_t^2\lambda_2(1-\xi)(1-\zeta)\zeta Q^{-1-\zeta}+q_th_t\lambda_3
\end{aligned}
\tag{59}
$$

After eliminating h_t, λ_t, q_t using (54), and substituting the steady state values above, we finally arrive at the *correct quadratic approximation to the single-period utility* in the expected intertemporal utility function (38):

$$
\begin{aligned}
&-\frac{\kappa}{2\alpha}\frac{Y^{1+\phi}}{A^{1+\phi}}\left[\frac{\sigma}{1-h_C}(y_t-z_t)^2+\phi(\alpha+\lambda_5(1+\phi))y_t^2\right.\\
&-2(1+\phi)(\alpha+\lambda_5(1+\phi))y_ta_t+2\lambda_5\frac{\sigma}{1-h_C}(y_t-z_t)y_t\\
&\left.-\lambda_5\frac{\sigma(\sigma+1)}{(1-h_C)^2}(y_t-z_t)^2+\frac{\xi\zeta}{(1-\xi)(1-\beta\xi)}(\alpha+(1+\phi)\lambda_5)\pi_t^2\right]
\end{aligned}
\tag{60}
$$

4.5 The Social Planner's Problem

The Social Planner can be regarded as maximizing (21) viewing all agents as identical, and so can set $C_{it} = C_t, N_{it} = N_t$, subject to the constraint $C_t = Y_t = A_tN_t$. The social planner chooses a trajectory for output which satisfies the first-order condition

$$[C_t - h_C C_{t-1}]^{-\sigma} - h_C\beta[C_{t+1} - h_C C_t]^{-\sigma} = \kappa\frac{Y_t^{\phi}}{A_t^{1+\phi}} \tag{61}$$

The efficient steady-state level of output $Y_{t+1} = Y_t = Y_{t-1} = Y^*$, say, is therefore given by

$$(Y^*)^{\phi+\sigma} = \frac{(1-h_C\beta)A^{1+\phi}}{\kappa(1-h_C)^{\sigma}} \tag{62}$$

We can now examine the inefficiency of the zero-inflation steady state. From (51) the zero-inflation steady state output in the Ramsey problem is given by $Y = Y^R$ where

$$(Y^R)^{\phi+\sigma} = \frac{\left(1-\frac{1}{\zeta}\right)\left(1-\frac{1}{\eta}\right)A^{1+\phi}}{\kappa(1-h_C)^{\sigma}(1-h_N)^{\phi}} \tag{63}$$

It is easy to check that this is exactly the same steady-state level as that of the flexi-price economy where firms set prices optimally at every period. Comparing (62) and (63) we have the result first obtained by Choudhary and Levine (2005):

Result 2:

The natural level of output, Y^R, is below the efficient level, Y^*, if and only if

$$\alpha \equiv \left(1 - \frac{1}{\zeta}\right)\left(1 - \frac{1}{\eta}\right) < 1 - h_C \beta \tag{64}$$

In the case where there is no habit persistence in consumption, $h_C = 0$, then (64) always holds. In this case market power in the output and labour markets captured by the elasticities η, ζ respectively drive the natural rate of output below the efficient level. If habit persistence in consumption is sufficiently high, then (64) does not hold and the natural rate of output and employment proportional are then too high compared with the efficient outcome and people are working too much. Is there empirical support that (64) holds? Terms $\left(1 - \frac{1}{\zeta}\right)$ and $\left(1 - \frac{1}{\eta}\right)$ are the inverses of mark-ups over marginal costs in the output and labour markets respectively. A plausible upper bound on these mark-ups is 20% so $\alpha = \left(1 - \frac{1}{\zeta}\right)\left(1 - \frac{1}{\eta}\right) > \frac{1}{1.2^2}$. A condition on h_C for (64) to hold is therefore $h_C\beta < 0.306$. Most empirical estimates of habit in a quarterly model are in the range $h_C = [0.5, 0.9]$ which would see these condition not holding (see, for example, Smets and Wouters (2003)).

4.6 The Small Distortion Case

The small distortion case assumes that the inefficiency of the zero-inflation steady state about which we have linearized is approximately efficient. From result 2 this implies that $1-\beta h_C-\alpha$ is small. We are now in a position to examine the nature of this 'approximation to an approximation' by examining correctly quadratified single-period utility (60). From (52) we can see that the approximates means that λ_5 is small. An examination of (60) reveals that the small distortion case, which would omit all terms involving λ_5, is valid only if $\mid \lambda_5(1+\phi) \mid << \alpha$ or, using the definition of λ_5, only if

$$(1+\phi)\frac{\mid 1 - \beta h_C - \alpha \mid}{\frac{\sigma(1-h_C\beta)}{1-h_C} + \phi} << \alpha \tag{65}$$

Typical estimated parameter values are $\sigma = 3$ (with this value or higher being confirmed within other contexts as well), $\phi = 1.3$. With h_C at the mid-point of the range of estimates at $h_C = 0.7$ this gives the left-hand-side of (65) as 0.22 and the right-hand side as 0.69. Neglected terms are therefore of the order of one third of those retained.

5 Target Implementability and the Ramsey Inflation Rate

Although the previous section has solved the policymaker's Ramsey problem, in practice there is no guarantee that monetary policy will be implemented in this fashion. Firstly, the usual instrument of monetary policy is the interest rate, which is under the control of the Central Bank. Secondly, although an increasing number of central banks have become more independent, transparent and accountable, none satisfies all the criteria for these attributes. As a consequence there can be no certainty that central banks will adopt a fully optimal precommitment policy. What is more likely is that any precommitment is likely to be to some simple rule, such as feedback on inflation and the output gap. This is much more easily monitored than the fully optimal rule, and simulations by numerous authors have shown that the welfare losses from using precommitted simple rules are considerably less than those from optimal rules under no commitment.

There is one aspect of fully optimal rules that appears not to be in dispute. Since the steady state setting of the optimal rule is potentially easily monitored, there is no reason why the central bank cannot commit to it, so the only issue with regard to precommitment is its stabilization aspect. It is therefore at this point that that use of the quadratic approximation to utility is appropriate.

Svensson (2005) suggests that central banks engage in 'forecast targeting', so that in effect they set targets for a set of variables of which inflation is but one. In the context of our quadratic approximations we can interpret these targets as 'bliss points', provided that the period t quadratic approximation achieves a maximum at these. This is also related to operational transparency of central banks, 'defined as the extent to which the monetary control errors are disclosed to the private sector' Geraats (2002). Faust and Svensson (2001) show that social welfare is improved with greater operational transparency provided that the output target is the natural rate, and not an 'ambitious' one.

Definition: A period-t welfare function is **Target-Implementable** if it is a maximum at its 'bliss points'.

This leads to the following, which follows directly from Result 1:

Result 2: The Ramsey solution is **Target-Implementable** if the quadratic approximation to the Lagrangian is negative semi-definite[5].

We now turn to the issue of whether the zero-inflation steady-state of the system does indeed constitute a maximum to the Ramsey problem. In the absence of habit, $h_C = 0$, it turns out that the sufficient condition of Result 1 is indeed satisfied; after some further effort (and subtracting an appropriate term in a_t^2), (60) further reduces to

$$\frac{-\kappa Y^{\phi+1}}{2\alpha A^{1+\phi}}(\phi + \sigma\alpha + 1 - \alpha)\left[(y_t^2 - \frac{1+\phi}{\sigma+\phi}a_t)^2 + \frac{\zeta\xi}{(1-\xi)(1-\beta\xi)(\sigma+\phi)}\pi_t^2\right] \tag{66}$$

This is clearly negative definite, so that the zero-inflation equilibrium is indeed an optimum in the absence of habit.

From (66) we note that the stochastic output target implied by this expression is $\frac{1+\phi}{\sigma+\phi}a_t$, as one would expect from first principles. We also note that this is a very similar expression to that derived by Benigno and Woodford (2004), although their setup is slightly different; it is easiest to think of their model as each firm only employing one type of labour, so that each agent supplies a different quantity of labour. Price dispersion then plays a role as a consequence of the policymaker maximizing over the average utility function. A comparison of their efforts to obtain a quadratic approximation shows that it is much more laborious than the Lagrangian method adopted here.

Now suppose that $h_C > 0$ and that (64) is not satisfied (as would seem plausible). So far we have not yet demonstrated whether the natural rate as calculated, with zero inflation, is actually the steady state for the Ramsey problem. To check this, we need either to solve the corresponding Riccati equation or to check the sufficient conditions of Target-Implementability. If the sufficient conditions of the latter are not satisfied, then checking the steady state Riccati matrix will not yield analytic results. This is because the equation governing it is highly nonlinear, and in addition the matrix is of dimension 2, so analytic solutions will not in general be found.

We therefore focus on Target-Implementability, and determine what conditions on the underlying parameters are required for (60) to be negative semi-definite. In order to reduce

[5]This means that the period-t utility in the LQ-approximation can be written as a weighted sum of squares of linear terms, with all weights negative.

the algebraic burden, we make the relatively innocuous approximation $\beta = 1$, since most quarterly models would assume a value of the order of 0.99. Before stating the result, we recall that the solution for the steady state level of output provided in Section 3 satisfies the first-order conditions when inflation is zero.

Proposition

A sufficient condition for the Ramsey problem with habit in consumption to have a *non-inflationary* steady state with a natural rate of output (63) and to be Target-Implementable is that (i) $\sigma > 1$ (ii) $\phi\sigma^2 > \phi + \sigma$.

Proof: See Appendix.

Using typical estimated parameter values discussed above both of these sufficient conditions are easily satisfied.

6 Conclusions

We introduced LQ approximations by expanding the welfare function for the Ramsey problem about the steady state zero-inflation efficient level. We remarked that linearization of the dynamics is only valid about this steady state if there is a tax/subsidy that ensures that the steady state of the Ramsey problem is itself efficient.

We have shown that a procedure proposed by Benigno and Woodford (2003) for large underlying distortions in the economy can be more easily implemented through a second-order approximation of the Lagrangian used to compute the ex ante optimal policy with commitment (the Ramsey problem). We have also examined in detail the LQ approximation of a particular Dynamic Stochastic General Equilibrium (DSGE) model, pointing out the necessary and sufficient conditions for a maximum. We show the limitations of the 'small distortions' approximation to an approximation both generally and in the context of this particular model.

We then defined the notion of Target-Implementability, which we argue is desirable for the transparency of stabilization policy in that the objectives in the loss function can be formulated in terms of bliss points. We assessed Target-Implementability for the particular case of habit inducing excessive labour input compared with the efficient level. We showed that the condition for both Target-Implementability and a zero inflation steady state to the Ramsey problem is that the quadratic approximation of the single-period utility about such a steady state is negative semi-definite. We obtain a sufficient condition for negative semi-definiteness and we find it indeed is satisfied for all plausible parameter values.

A Example 2: Equivalence of the Benigno-Woodford and Hamiltonian Procedures

A.1 The Benigno-Woodford Procedure

To find $(\bar{\pi}, \bar{x})$ and h first write

$$\begin{aligned}(x_t - x^*)^2 + \pi_t^2 &\equiv (x_t - \bar{x} + \bar{x} - x^*)^2 + (\pi_t - \bar{\pi} + \bar{\pi})^2 \\ &= (x_t - \bar{x})^2 + 2(x_t - \bar{x})(\bar{x} - x^*) + (\pi_t - \bar{\pi})^2 + 2\bar{\pi}(\pi_t - \bar{\pi}) \\ &+ \text{constant terms} \end{aligned} \tag{A.1}$$

Then (15) holds iff at each time t

$$\begin{aligned}\theta(x_t - \bar{x})^2 &+ \phi(\pi_t - \bar{\pi})^2] \equiv (x_t - \bar{x})^2 + 2(x_t - \bar{x})(\bar{x} - x^*) + (\pi_t - \bar{\pi})^2 + 2\bar{\pi}(\pi_t - \bar{\pi}) \\ &+ h\left(\frac{\beta}{\lambda}(\pi_t - \bar{\pi}) - (\pi_t - \bar{\pi}) - a(x_t - \bar{x}) + b(x_t - \bar{x})^2\right)\end{aligned} \tag{A.2}$$

Equating quadratic and linear terms we arrive at

$$\theta = 1 + hb \tag{A.3}$$

$$\phi = 1 \tag{A.4}$$

$$2(\bar{x} - x^*) - ha = 0 \tag{A.5}$$

$$2\bar{\pi} + h\left(\frac{\beta}{\lambda} - 1\right) = 0 \tag{A.6}$$

Then together with the condition for $(\bar{\pi}, \bar{x})$ to be a steady state:

$$(\beta - 1)\bar{\pi} - f(\bar{x}) = 0 \tag{A.7}$$

we have 5 equations to solve to θ, ϕ, h, $\bar{\pi}$ and $\bar{x}$. The solution is

$$h = -\frac{2(x^* - \bar{x})}{a} < 0 \text{ if } x^* > \bar{x} \tag{A.8}$$

$$\bar{\pi} = \left(1 - \frac{\beta}{\lambda}\right)\frac{h}{2} > 0 \text{ iff } \beta > \lambda \text{ and } x^* > \bar{x} \tag{A.9}$$

$$\theta = 1 + hb < 1 \text{ if } x^* > \bar{x} \tag{A.10}$$

where $\bar{x}$ is the solution to

$$(\beta - 1)\left(1 - \frac{\beta}{\lambda}\right)\frac{(x^* - \bar{x})}{a} + f(\bar{x}) = 0 \tag{A.11}$$

If the policymaker adopts the same discount factor as the private sector, then $\lambda = \beta$ and $\bar{\pi} = 0$; that is the steady state is the same deterministic non-inflationary steady state x, where $f(x) = 0$, we chose for (17). Let us assume that indeed $\lambda = \beta$. Then comparing the BW procedure with the standard linear-quadratic approximation discussed at the beginning of this section, we see that the later is only a good approximation if $(x^* - \bar{x})$ or b are small. In the former case this implies that the output target is close to the non-inflationary stated state of x_t, whilst in the latter case the Phillips curve is nearly linear. If neither of these conditions apply then the BW procedure must be used.

A.2 The Hamiltonian Procedure

We now show that the LQ procedure of BW is equivalent to a rather simpler one. Consider the deterministic problem to choose at $t = 0$ a trajectory $\{\pi_t\}$ so as to minimize

$$\left[\sum_{t=0}^{\infty} \lambda^t \left[(x_t - x^*)^2 + \pi_t^2\right]\right] \tag{A.12}$$

subject to

$$\pi_t = \beta\pi_{t+1} + f(x_t) \tag{A.13}$$

To solve this problem we minimize a Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \lambda^t \left[(x_t - x^*)^2 + \pi_t^2 + \mu_t(\pi_t - \beta\pi_{t+1} - f(x_t))\right] \tag{A.14}$$

with respect to $\{\pi_t\}$, $\{x_t\}$ and the Lagrangian multiplier $\{\mu_t\}$. This is the deterministic component of our original non-linear optimization problem available to the policymaker if she can commit. The first-order conditions for this problem are

$$2(x_t - x^*) - \mu_t f'(x_t) = 0 \tag{A.15}$$

$$2\pi_t + \mu_t - \frac{\beta}{\lambda}\mu_{t-1} = 0 \tag{A.16}$$

$$\pi_t - \beta\pi_{t+1} - f(x_t) = 0 \tag{A.17}$$

This system has a steady state (x,π,μ) at

$$
\begin{aligned}
2(x-x^*)-\mu f'(x) &= 0 && \text{(A.18)}\\
2\pi+\mu\left(1-\frac{\beta}{\lambda}\right) &= 0 && \text{(A.19)}\\
(1-\beta)\pi-f(x) &= 0 && \text{(A.20)}
\end{aligned}
$$

Comparing (A.18) to (A.20) with (A.5) to (A.7) and noting that $a=f'(\bar{x})$ in (A.5) it is immediately apparent that $(x,\pi,\mu)=(\bar{x},\bar{\pi},h)$ found in the BW procedure. Then the modified loss function (**??**) is a second-order Taylor series approximation to the Lagrangian (A.14) evaluated at the steady state of the optimal commitment solution $\mu_t=\mu$ in the vicinity of (x,π).

B Proof of Proposition

Firstly, we require the coefficient of π_t^2 inside the brackets of (60), $\alpha+(1+\phi)\lambda_5$, to be positive. A little calculation shows that (with $\alpha>1-h_C$) this term is greater than $1-h_C$ provided that $\sigma>1$. Ignoring the shock term a_t, if we now consider the remaining terms as a quadratic function of y_t and y_t-z_t, then this quadratic will always be positive provided that (a) $\alpha+(1+\phi)\lambda_5>0$, (b) $\frac{\sigma}{1-h_C}(1-\frac{\lambda_5(1+\sigma)}{1-h_C})>0$ and (c) $\phi(\alpha+(1+\phi)\lambda_5)\frac{\sigma}{1-h_C}(1-\frac{\lambda_5(1+\sigma)}{1-h_C})-\frac{\lambda_5^2\sigma^2}{(1-h_C)^2}>0$. (a) has already been shown, and it is easy to show that the left hand side of (b) is greater than $\sigma/(1-h_C)^2$. After some manipulation we can show that after multiplying (c) through by $(1-h_C)^2$ the left hand side becomes

$$
\begin{aligned}
&(\phi^3-\phi-\sigma)(1-h_C)^2+2\alpha(1-h_C)(\sigma\phi^2+\phi+\sigma)+(\phi\sigma^2-\phi-\sigma)\alpha^2\\
&>(\phi^3+\sigma\phi^2)(1-h_C)^2+\alpha(1-h_C)(\sigma\phi^2+\phi+\sigma)+(\phi\sigma^2-\phi-\sigma)\alpha^2 \qquad \text{(B.21)}
\end{aligned}
$$

where the inequality holds when $\alpha>1-h_C$. Thus the sufficient condition $\phi\sigma^2-\phi-\sigma>0$ is likely to be considerably more stringent a condition than is required.

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