

# Natural volatility, welfare and taxation

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Cyclical components are analytically computed in a theoretical model of stochastic endogenous fluctuations and growth. Volatility is shown to depend on the speed of convergence of the cyclical component, the expected length of a cycle and on the altitude of the slump. Taxes affect these channels and can therefore explain cross-country differences and breaks over time in volatility. With exogenous sources of fluctuations, a special case of our model, decentralized factor allocation is efficient. With endogenous fluctuations and growth, decentralized factor allocation is inefficient and (time-invariant) taxes can (de-) stabilize the economy. No unambiguous link exists between volatility and welfare.

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## 1 Introduction

There is a considerable heterogeneity across OECD countries in the variance of annual GDP growth rates. This variance ranges from 25% and 15% for Greece and Japan in the 1961 to 1983 period to 1.7% and 1.3% for France and Italy for 1984 to 2003. Empirical studies show further that countries have breaks in their variance of growth rates over time (e.g. Kim and Nelson, 1999; McConnell and Perez-Quiros, 2000; Stock and Watson, 2003). Do these

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differences in volatility have only external causes such as terms of trade shocks, monetary or exogenous productivity shocks? Or is growth volatility of a country an endogenous, natural phenomenon of any growing economy and thereby also a function of various fundamentals of the economy under consideration, including economic policy? Stock and Watson (2003), surveying the literature on the “big moderation”, attribute (roughly and with caveats) one quarter of the moderation in volatility in the US to improved policy, one quarter to good luck (lower volatility of productivity and commodity price shocks) and 50% to “unknown forms of good luck”.

This paper undertakes a theoretical and analytical investigation of determinants of economic volatility. It is argued that cross-country differences in constant and time-invariant tax rates are one possible source behind observed cross-country differences in volatility. Similarly, a change in tax legislation at a given point in time can in principle explain increases or decreases of volatility.

This analysis is based on the view that volatility of a country is something natural, inherently linked to its growth process. Volatility is just as endogenous as is the GDP growth rate. Volatility and long-run growth result primarily from the introduction of new technologies. “Lower volatility of productivity” or “other unknown forms of good luck” can therefore be explained by more fundamental changes in an economy. As both long-run growth and short-run volatility are endogenous and therefore react to changes in policy, we can analyze to what extent tax rates affect volatility and growth at the same time or independently of each other. Analysing long-run growth and short-run volatility jointly is important as understanding e.g. the “big moderation” in the US seems to require a break in volatility without a break in the growth trend (McConnell and Perez-Quiros, 2000).

The model we employ is part of a small but rapidly growing literature that integrates endogenous short-run fluctuations with endogenous long-run growth (e.g. Bental and Peled, 1996; Matsuyama, 1999; Wälde, 1999, 2005; Francois and Lloyd-Ellis, 2003; Maliar and Maliar, 2004, Phillips and Wrase, 2005). These papers share the view that intentional investment into R&D can not only explain long-run growth but also short-run fluctuations - without invoking exogenous disturbances to the economy. More productive technologies increase TFP in a discrete way, similar to a step function, and not smoothly and continuously as in standard models.<sup>2</sup> As a consequence, new technologies cause both short-run booms (due to the discrete increase) and long-run growth.<sup>3</sup> As our model has an explicit stochastic

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<sup>2</sup>This step-wise increase of productivity is well-known from quality ladder models. See e.g. Grossman and Helpman (1991), Aghion and Howitt (1992), Aghion et al. (2001) or Francois and Roberts (2003).

<sup>3</sup>Due to the explicit modelling of R&D processes, these models can be viewed to represent industrialized

foundation, it shares Beveridge and Nelson's (1981) econometric view that trend and cycle are driven by the same shock, i.e. (here) stochastic jumps in the technological frontier. We will show below that due to the vintage-capital structure we employ here, there are *no* jumps in TFP in our model as all technological change is embodied. This improves features of e.g. Aghion and Howitt (1992) or Francois and Lloyd-Ellis (2003).

We relate our analysis of natural volatility to taxation as there is a considerable heterogeneity in tax systems across countries and over time. For the US, two major tax reforms, the Tax Reform Act of 1986 and the Economic Recovery Tax Act of 1981 (see e.g. Auerbach and Slemrod, 1997), took place around the point in time where the break in GDP volatility is usually identified (between the 4th quarter of 1982 and 3rd quarter of 1985, according to Stock and Watson, 2003). It is therefore natural to ask whether tax reforms or cross-country differences in tax systems are candidates for understanding differences in volatility. It is generally accepted that taxes can affect the growth rate of a country or its natural rate of unemployment - they could therefore also affect its natural amount of volatility.<sup>4</sup>

One contribution of our paper is the derivation of two analytical measures of volatility in a model characterized by "standard" properties: infinite planning horizon of the representative agent, standard intertemporal optimization decisions concerning savings and investment under risk aversion, uncertainty from properties of technological progress and perfect competition for all production processes. These two measures are the variance of the growth rate of the economy, a widely used measure in empirical regression work, and the coefficient of variation for cyclical components of time series, similar to those used in the RBC literature. It turns out that the variance of the growth rate does not - due to its complexity - lend itself to an intuitive theoretical analysis. Cyclical components have very simple moments, however, that reveal insightful relationships between model parameters and volatility.

Analytical measures for volatility can be obtained by assuming a simple parameter restriction. Analyzing the behaviour of an economy for restrictions of this type has turned out to be very useful (e.g. Long and Plosser, 1983; Xie, 1991; Benhabib and Rustichini, 1994;

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economies. Aghion et al. (1999) use an AK-type economy which exhibits endogenous growth and fluctuations as well. Their model exhibits strong imperfections on financial markets and can therefore be seen as an analysis of developing countries. Jovanovic (2006) also presents a model with growth and cycles. He links (unpredictable) skill needs to technological progress and focuses on the (a)symmetric properties of cycles. For a broader background, see Gancia and Zilibotti (2005, ch. 7). An early contribution with similar properties as Matsuyama (1999) is Zilibotti (1995).

<sup>4</sup>First econometric evidence shows that about two third of the variation of output volatility in OECD countries can be explained by various fundamentals including tax policy (Posch, 2006). An alternative approach to our theoretical inquiry here is provided by the quantitative analysis of Arias et al. (2006).

Wälde, 2005). When looking at our results, it becomes clear that this restriction has no major economic implications.<sup>5</sup>

The coefficient of variation shows that volatility is affected through three channels: The speed of convergence, the expected length of a cycle and the degree how strongly cyclical components are thrown back once a new technology arrives. All of these three channels can be easily related to properties of transitional paths towards some steady state. As taxes affect transitional paths of various economic variables, taxation affects volatility. For the equilibrium we analyze, taxes on factor rewards and investment goods increase volatility, taxes on R&D and wealth have a stabilizing effect, a tax on consumption goods is neutral. We show then how one change of otherwise constant taxes can be used to build intuition about potential sources of the “big moderation”.

We also ask whether higher or lower volatility should give rise to policy concerns. One possible answer is a clear ‘no’. The RBC approach is built (at least initially) on the belief that agents adjust optimally to a fluctuating world where factor allocations are efficient. The present paper argues that volatility per se is not problematic for welfare indeed - as long as one believes that the engines of growth of an economy work under the absence of any imperfections. If, however, one believes that (empirically speaking: at least to some extent) fluctuations in an economy are the result of the same type of technological progress that causes long-run growth, fluctuating economies are as efficient or inefficient as the process that drives long-run growth. Hence, taking the lessons from the “new” growth theory seriously where it might be difficult in R&D based models to justify that endogenous technological progress comes along without any externalities, fluctuations go hand in hand with imperfections. This is true even in our setup where all firms operate under perfect competition, including R&D firms.<sup>6</sup>

Our welfare results are based on the value function of the representative household. We obtain a closed-form expression of the value function also due to our simple parameter restriction. In deriving it, we obtain a deterministic differential equation that describes how the economy evolves in an expected sense, i.e. how expected instantaneous utility evolves for  $\tau > t$ , where  $t$  is today. This differential equation shows that our economy behaves in

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<sup>5</sup>From a modeling perspective, the present paper extends the model developed in Wälde (2005) for various tax rates and the government sector. The methods of Garcia and Griego (1994), on which most of our results here are based, were not used in Wälde (2005).

<sup>6</sup>There are by now various papers that stress that R&D and perfect competition does not contradict each other. The first paper seems to be Funk (1996). Later work includes e.g. Boldrin and Levine (2002, 2004) or Hellwig and Irmen (2001).

this expected sense exactly as a deterministic Solow growth economy behaves with a fixed saving rate. Intuitively speaking, our stochastic economy turns out to be a Solow growth economy where labour productivity increases at (endogenous and optimally chosen) random points in time by discrete amounts.

Returning to the effect of taxes, our welfare analysis shows that taxes on investment goods and R&D directly affect the source of volatility and growth, i.e. the portfolio choice between capital accumulation and R&D, and can therefore be used to internalize externalities. All other taxes (on factor rewards, consumption and wealth) are welfare reducing, given that they are used for some exogenous government expenditure (which, for simplicity, is modelled to have no welfare or productivity effect). When we look at the effects of taxes on volatility and welfare jointly, it turns out that stabilizing an economy is not necessarily welfare increasing. Increasing a tax on wealth or factor income reduces welfare, but the tax on factor income increases volatility while the tax on wealth reduces volatility. The objective of government intervention should be to internalize external effects, as in standard public finance approaches, but not to stabilize the economy. The efficient factor allocation would then be characterized by a certain corresponding amount of volatility. The negative causal link from volatility to welfare is therefore opened up under endogenous volatility. Volatility and welfare are only (positively or negatively) correlated and more volatility can mean higher welfare.

## 2 The model

The model will be presented in three parts: Technologies, the government and consumers. As the technological setup of our economy is close to the one in Wälde (2005), the first part will be relatively brief. The introduction of government activities and the implications for household behavior are new and will be presented in more detail.

### 2.1 Technologies

Technological progress is labour augmenting and embodied in capital. All capital goods can be identified by a number denoting their date of manufacture and therefore their vintage. A capital good  $K_j$  of vintage  $j$  allows workers to produce with a labour productivity  $A^j$ , where  $A > 1$  is a constant parameter. Hence, a more modern vintage  $j + 1$  implies a labour productivity that is  $A$  times higher than labour productivity of vintage  $j$ . The corresponding production function reads  $Y_j = K_j^\alpha (A^j L_j)^{1-\alpha}$ , where the amount of labour allocated to that

vintage is denoted by  $L_j$  and  $0 < \alpha < 1$  is the output elasticity of capital.

R&D is a risky activity. This is modelled by the Poisson process  $q$  where the probability per unit of time  $dt$  of an innovation, i.e. of successful R&D, is given by  $\lambda dt$ , where  $\lambda$  is the arrival rate of  $q$ . At the level of an individual R&D firm  $f$ , there are constant returns to scale and the firm arrival rate is  $\lambda_f = D^{-1}h(R/D)R_f$ , where  $D$  captures the “difficulty” of doing R&D,  $h(\cdot)$  is an externality and  $R_f$  are resources used by the firm. The difficulty function  $D$  and the externality  $h(\cdot)$  are taken as given. As firm-level Poisson processes  $q_f$  can be added up, we obtain

$$\lambda = \frac{R}{D}h\left(\frac{R}{D}\right) \equiv \left(\frac{R}{D}\right)^{1-\gamma}, \quad 0 < \gamma < 1, \quad (1)$$

at the sectoral level where  $h(\cdot)$  implies decreasing returns to scale. The second equality implicitly defines the functional form of  $h(\cdot)$ .

The exogenous function  $D$  captures the difficulty to make an invention. Following the arguments in Segerstrom (1998), an economy that discovered already many innovations needs to put more effort into a new innovation if this innovation is to come at the same rate  $\lambda$ . While the amount of innovations in the past can be measured in different ways, we simply capture it by the tax-independent current size  $K_*^{obs}$  of the capital stock of the economy,

$$D \equiv D_0 K_*^{obs}, \quad D_0 > 0. \quad (2)$$

This measure of the capital stock will be defined in (12).

R&D resources  $R$  are used to develop a capital good that yields a higher labour productivity than existing capital goods. The currently most advanced vintage is denoted by  $q$  and implies a labour productivity of  $A^q$ . The outcome of successful R&D is a first prototype of a production unit of size  $\kappa$  (whose implied labour productivity is  $A^{q+1}$ ). The production of  $\kappa$  through R&D distinguishes our approach from standard modeling of R&D where successful R&D is argued to lead to a blueprint only. As seems to be common in many cases (Rosenberg, 1994), only the development of a first “pioneer plant” that can be used for production characterizes success of research. We can capture this creation of the first production unit of vintage  $q + 1$  by noting that the increment  $dq$  of the Poisson process  $q$  can either be 0 or 1. As successful research means  $dq = 1$ , we can write

$$dK_{q+1} = \kappa dq. \quad (3)$$

The “importance” of the prototype can be argued to increase in the amount of time and resources  $R$  spent on developing  $\kappa$ . Longer research could imply a larger prototype. If capital

goods were not perfect substitutes as here (see e.g. (9) below), longer research would imply that the prototype is more valuable at the moment of development as the old capital stock will then be larger. We capture these aspects in a simple and tractable way by keeping  $\kappa$  proportional to the tax-independent size  $K_*^{obs}$  of the total capital stock,

$$\kappa \equiv \kappa_0 K_*^{obs}, \quad 0 < \kappa_0 \ll 1. \quad (4)$$

When resources are allocated to capital accumulation, the capital stock of vintage  $j$  increases if investment in vintage  $j$  exceeds depreciation  $\delta$ ,

$$dK_j = \{I_j - \delta K_j\} dt, \quad j = 0, \dots, q. \quad (5)$$

In contrast to R&D, this is a deterministic process as capital accumulation simply means replicating existing machines.

Before we continue with the description of the model, some equilibrium properties are useful as they simplify presentation of the government and household part. Each vintage of capital allows to produce the same type of good,

$$\sum_{j=0}^q Y_j = Y = C + I + R + G. \quad (6)$$

Aggregate output  $Y$  is used for producing consumption goods  $C$ , investment goods  $I$ , as an input  $R$  for doing R&D and for government expenditures  $G$ . The quantities  $C$ ,  $I$  and  $R$  stand for *net* resources used for these activities, i.e. after taxation. All activities in the economy take place under perfect competition. The producer prices of the production, consumption, investment and research good will therefore be identical,

$$p_Y = p_C = p_I = p_R. \quad (7)$$

Total exogenous labour supply in this economy is  $L$ . Allowing labour to be mobile across all vintages such that wage rates equalize and assuming full employment,  $\sum_{j=0}^q L_j = L$ , total output of the economy can be represented by a simple Cobb-Douglas production function,

$$Y = K^\alpha L^{1-\alpha}, \quad (8)$$

where vintage specific capital stocks have been aggregated to an aggregate capital index  $K$ ,

$$K = K_0 + BK_1 + \dots + B^q K_q = \sum_{j=0}^q B^j K_j, \quad B \equiv A^{\frac{1-\alpha}{\alpha}}. \quad (9)$$

This index can be thought of as counting the “number of machines” of vintage 0 that would be required to produce the same output  $Y$  as with the current mix of vintages.

The evolution of the capital index  $K$  follows from (3) and (5) by applying the change of variable formula (CVF)<sup>7</sup> to (9),

$$dK = \{B^q I - \delta K\} dt + B^{q+1} \kappa dq. \quad (10)$$

The capital index increases continuously as a function of effective investment  $B^q I$  minus depreciation. When an innovation takes place, the capital index increases by  $B^{q+1} \kappa$ .

## 2.2 Government

The government can levy taxes on factor income ( $\tau_F$ ), wealth ( $\tau_W$ ), consumption expenditure ( $\tau_C$ ), investment expenditure ( $\tau_I$ ) and R&D expenditure ( $\tau_R$ ). A positive tax implies a real decrease in income or an increase in the effective price (consumer price), whereas a negative tax is a subsidy. The government uses all tax income (and does not save or run a debt) to provide basic government services  $G$ . In order to focus on the effects of taxation from government expenditures, we assume that government expenditure does not affect household utility or production possibilities of the economy.

Producer prices by (7) are identical for all three production processes. When consumption and investment goods  $C$  and  $I$  or research services  $R$  are sold, they are taxed differently such that consumer prices are  $(1 + \tau_C) p_C$ ,  $(1 + \tau_I) p_I$ ,  $(1 + \tau_R) p_R$ . In order to rule out arbitrage between different types of goods, we assume that once a unit of production is assigned for a special purpose, it is useless for other purposes: once a consumption good is acquired, it cannot be used for e.g. investment purposes.

Taxes that increase the producer price have no theoretical upper bound. A 300% tax on the consumption good would imply that 3/4 of the price are taxes going to the state and 1/4 goes to the producer. Their lower bound is clearly  $-100\%$ , where a good would be for free for the purchaser. The upper bound for taxes on income is 100% (instant confiscation of income), while there is no lower bound. Hence,  $-1 < \tau_C, \tau_I, \tau_R$  and  $\tau_F, \tau_W < 1$ .

Our capital stock index  $K$  in (9) measures the size of the capital stock in units of vintage 0. Measured in units of vintage  $q$ , its size is  $B^{-q} K$ . This is also the value of the entire capital stock in pre-tax units of the consumption good, as the relative pre-tax prices are unity from (7). Measuring wealth in after-tax prices, i.e. in “purchasing power” terms, the price of the

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<sup>7</sup>In models with Brownian motion as a source of uncertainty, the “rules” for computing differentials are based on Ito’s Lemma. In the presence of Poisson processes, the CVF is the appropriate “rule”. See e.g. Garcia and Griego (1994) and Sennewald (2006) for a rigorous background and Sennewald and Wälde (2006) for an introduction.



capital good increases by the tax  $\tau_I$  and the price to be paid for one unit of the consumption good increases by  $\tau_C$ . Hence, total wealth in the economy is given by

$$K^{obs} = La = \frac{1 + \tau_I}{1 + \tau_C} B^{-q} K, \quad (11)$$

where  $a$  is wealth of the representative household. The tax-independent measure of the capital stock, used in (2) and (4), can then be defined by

$$K_*^{obs} \equiv \frac{1 + \tau_C}{1 + \tau_I} K^{obs}. \quad (12)$$

## 2.3 Households

The economy is populated by a finite number of sufficiently small representative households. They maximize expected utility  $U(t)$ , given by the “sum” of instantaneous utilities  $u(\cdot)$  resulting from consumption flows  $c(\tau)$ , discounted at the rate of time preference  $\rho$ ,

$$U(t) = E_t \int_t^\infty e^{-\rho[\tau-t]} u(c(\tau)) d\tau, \quad (13)$$

where instantaneous utility  $u(\cdot)$  is characterized by constant relative risk aversion,<sup>8</sup>

$$u(c(\tau)) = \frac{c(\tau)^{1-\sigma}}{1-\sigma}, \quad \sigma > 0. \quad (14)$$

The budget constraint reflects investment possibilities in this economy and the impact of tax policy and shows how real wealth  $a$  evolves over time. Households can invest in a risky asset by financing R&D and in an (instantaneously) riskless asset by accumulating capital. We measure wealth in units of the consumption good, priced at consumer prices. The budget constraint can intuitively best be understood by starting from (A.13) in the appendix,

$$\begin{aligned} da = & \left\{ \frac{1 - \tau_F}{1 + \tau_C} (\sum_{j=0}^{q+1} w_j^K k_j / p_C + w / p_C) - c - \frac{1 + \tau_R}{1 + \tau_C} i \right\} dt \\ & - \left\{ \frac{1 - \tau_F}{1 + \tau_I} \delta + \tau_W \right\} a dt + \left\{ \frac{1 + \tau_I}{1 + \tau_C} \kappa \frac{i}{R} - \frac{B - 1}{B} a \right\} dq. \end{aligned}$$

Nominal gross capital income  $\sum_{j=0}^{q+1} w_j^K k_j$  from all vintages  $j$  is taxed at  $\tau_F$ , yielding net capital income. Dividing by the consumer price  $(1 + \tau_C) p_C$  of the consumption good gives real net capital income in units of the consumption good. The same reasoning applies to labour income  $w$ , consumption  $c$  and investment  $i$  into R&D. The expression on the first line therefore captures the increase in wealth  $a$ , measured in units of the consumption good

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<sup>8</sup>For analytical convenience and readability, we neglect the term  $-(1 - \sigma)^{-1}$ , which is sometimes included in the instantaneous utility function.

at consumer prices. The first expression on the second line captures the wealth-reducing effect of the after-tax depreciation rate and of the tax on wealth. The tax rates  $\tau_F$  and  $\tau_I$  in front of the depreciation rate ensure that taxes are partly refunded i.e. only net (and not gross) investment will be taxed (cf. eqs. (A.14) and (A.2)). The second expression increases an individual's wealth in case of successful research by the "dividend payments" minus an economic depreciation term. Dividend payments at the household level are given by the share  $i/R$  of the successful research project the household financed times total dividend payments  $\frac{1+\tau_I}{1+\tau_C}\kappa$ . Dividend payments are determined by the size  $\kappa$  of the prototype times its after-tax price  $(1 + \tau_I) / (1 + \tau_C)$  in units of the consumption good.<sup>9</sup> The term  $1 + \tau_I$  implies that research yields not only a capital good (which would have a value of  $p_I$ ) but already an installed capital good (whose value is  $(1 + \tau_I)p_I$ ). Economic depreciation  $(B - 1) / B$  results from the vintage capital structure as the most advanced capital good has a relative price of unity (cf. (7)) and all other vintages then lose in value relative to the consumption good.

After some further steps and using the notation for values after taxation, the budget constraint simplifies to

$$da = \{r^*a + w^* - i^* - c\} dt + \{\kappa^*i/R - sa\} dq, \quad (15)$$

where  $i^* \equiv \frac{1+\tau_R}{1+\tau_C}i$ ,  $\kappa^* \equiv \frac{1+\tau_I}{1+\tau_C}\kappa$ ,  $s \equiv \frac{B-1}{B}$  and factor rewards are

$$r^* \equiv \frac{1 - \tau_F}{1 + \tau_I}r - \tau_W, \quad w^* \equiv \frac{1 - \tau_F}{1 + \tau_C}\frac{w}{p_C}, \quad r = B^q \frac{\partial Y}{\partial K} - \delta, \quad w = p_Y \frac{\partial Y}{\partial L}. \quad (16)$$

## 3 Natural volatility and growth

### 3.1 Equilibrium

Solving the model requires first order conditions for households for consumption and R&D expenditure. These two conditions, together with the aggregate capital accumulation constraint (10), the goods market equilibrium (6) and optimality conditions of perfectly competitive firms provides a system consisting of 6 equations that determines, given initial conditions, the time paths of  $K$ ,  $C$ ,  $R$ ,  $Y$ ,  $w$  and  $r$ .

Such a system can best be understood by introducing auxiliary variables that are similarly used in many other models as well: In the classic Solow growth model, capital per effective worker  $K / (AL)$  is shown to converge to a steady state and the analysis of e.g. convergence

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<sup>9</sup>We use the term dividend payments in a narrow sense, i.e. only for payments from successful R&D. Data on dividend payments would also include part of factor rewards  $r$  for capital.

can be separated from the analysis of long-run growth. In the present context, we define  $\hat{K}(\tau)$  and  $\hat{C}(\tau)$  as

$$\hat{K}(t) \equiv K(t)/A^{q(t)/\alpha}, \quad \hat{C}(t) \equiv C(t)/A^{q(t)} \quad (17)$$

which is almost identical to capital and consumption per effective worker as labor supply is constant here. These variables also allow us to separate the analysis of cyclical properties of the model from long-run growth. Most of the time, we will call  $\hat{K}(\tau)$  and  $\hat{C}(\tau)$  cyclical components of  $K(t)$  and  $C(t)$ , as  $A^{q(t)/\alpha}$  and  $A^{q(t)}$  will turn out to be the stochastic trend in our economy.

“Detrending” in (17) is undertaken by dividing by measures of the current technology level that differs between capital and consumption. This is due to the fact that  $K(t)$  is a capital index and not capital expressed in units of the consumption good. Capital measured in units of the consumption good would be detrended by  $A^{q(t)}$  as well. When detrending other endogenous variables by  $A^{q(t)}$  as well, these detrended variables turn out to be stationary and within a bounded range. Equilibrium properties can therefore best be illustrated by studying an equilibrium in cyclical components which consists of a system in 6 equations and 6 cyclical components as well.

### 3.2 An explicit solution

It would be interesting to analyze such a system in all generality. One would run the risk, however, of losing the big picture and rather be overwhelmed by many small results. We therefore restrict ourselves to a particular parameter set of the model that allows very sharp analytical results. Parameter restrictions have proven useful to derive equilibrium properties which otherwise would not be easily visible (e.g. Long and Plosser, 1983; Xie, 1991; Benhabib and Rustichini, 1994; Wälde, 2005).

**Theorem 1** *If the preference parameter of the utility function  $\sigma$  equals the output elasticity of the capital stock  $\alpha$ , that is*

$$\sigma = \alpha, \quad (18)$$

*we obtain a linear solution for consumption and research*

$$\hat{C} = \Psi \hat{K}, \quad \hat{R} = \Gamma \hat{K}, \quad (19)$$

*where  $\Psi$  and  $\Gamma$  denote constant parameters given by*

$$\Psi = \frac{1 + \tau_I}{1 + \tau_C} \left( \frac{\rho + \lambda(1 - (1 - s)\xi^{-\sigma})}{\sigma} + \frac{1 - \sigma}{\sigma} \left( \frac{1 - \tau_F}{1 + \tau_I} \delta + \tau_W \right) - \frac{1 + \tau_R}{1 + \tau_I} \Gamma \right), \quad (20)$$

$$\Gamma = \left( \frac{1 + \tau_I}{1 + \tau_R} \frac{\kappa_0}{D_0} \xi^{-\sigma} \right)^{\frac{1}{\gamma}} D_0. \quad (21)$$

The arrival rate is then constant and given by

$$\lambda = \left( \frac{1 + \tau_I}{1 + \tau_R} \frac{\kappa_0}{D_0} \xi^{-\sigma} \right)^{\frac{1-\gamma}{\gamma}}, \quad (22)$$

where we defined

$$\xi \equiv 1 - s + \kappa_0. \quad (23)$$

**Proof.** App. B.1.4. ■

We will assume in what follows that  $\xi < 1$  in (23), i.e. economic depreciation  $s$  due to the innovation is larger than the relative size of dividend payments  $\kappa_0$  from (4).

The parameter restriction  $\sigma = \alpha$  implies a relatively high intertemporal elasticity of substitution  $\sigma^{-1}$  of above unity. While there is supporting evidence (see Vissing-Jørgensen, 2002, and Gruber, 2006), the relevance of our results depends only on whether one believes that changes in  $\sigma$  will fundamentally change our insights. As will turn out further below, this is not the case.

### 3.3 Cyclical growth

Exploiting the implications of theorem 1, we can summarize general-equilibrium behaviour of agents in a way as simple as e.g. in the Solow growth model with exogenous growth and a constant saving rate, even though we have forward-looking agents and an uncertain environment. In terms of cyclical components, our economy follows (19) and (app. B.1.3)

$$d\hat{K} = \left\{ \hat{Y} - \hat{R} - \delta\hat{K} - \hat{C} - \hat{G} \right\} dt - \{1 - A^{-1}\xi\} \hat{K} dq \quad (24)$$

$$= \left\{ \frac{b_0}{\Psi^{1-\sigma}} \hat{K}^\alpha L^{1-\alpha} - \frac{b_1}{1-\sigma} \hat{K} \right\} dt - \{1 - A^{-1}\xi\} \hat{K} dq, \quad (25)$$

where with  $\Psi$  and  $\Gamma$  from (20) and (A.36),

$$b_0 \equiv \frac{1 - \tau_F}{1 + \tau_I} \Psi^{1-\sigma}, \quad (26)$$

$$\begin{aligned} b_1 &\equiv (1 - \sigma) \left( \frac{1 + \tau_C}{1 + \tau_I} \Psi + \frac{1 + \tau_R}{1 + \tau_I} \Gamma + \frac{1 - \tau_F}{1 + \tau_I} \delta + \tau_W \right) \\ &= \frac{1 - \sigma}{\sigma} \left( \rho + \lambda [1 - (1 - s) \xi^{-\sigma}] + \frac{1 - \tau_F}{1 + \tau_I} \delta + \tau_W \right). \end{aligned} \quad (27)$$

The differential equation (24) is the capital accumulation constraint (10), expressed for cyclical components and satisfying utility-maximizing behaviour of agents. Inserting (19)

and some further steps (app. B.2.1) give the one-dimensional stochastic differential equation (25) in  $\hat{K}$ .

Note that the expressions containing parameters  $b_0$  and  $b_1$  have an economic meaning: The first term represents cyclical output of this economy, reduced by taxation. This is visible from  $\hat{Y} = \hat{K}^\alpha L^{1-\alpha}$  (app. B.1.3). The  $b_1$  term represents resource allocation to R&D and consumption, in addition to physical capital depreciation, all corrected for taxation. As (25) shows,  $b_1$  also captures the speed of convergence of  $\hat{K}$  relative to its steady state. The differential equation (25) is illustrated in the following figure.

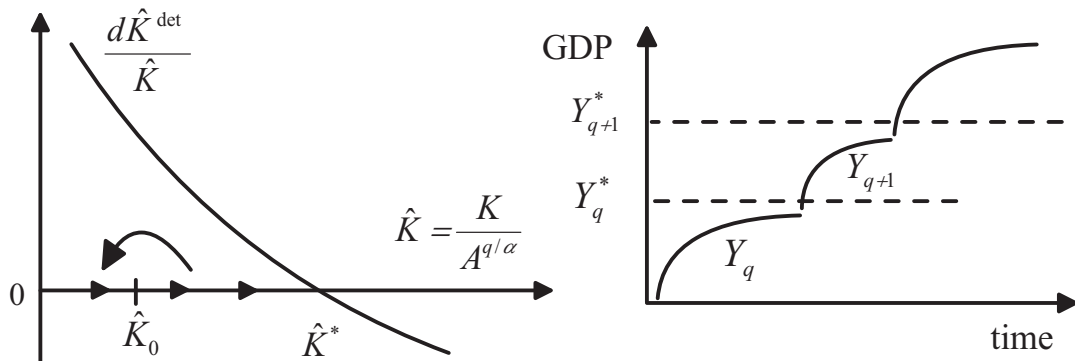


Figure 1: General equilibrium dynamics of the capital stock per effective worker and GDP growth cycles

The figure on the left plots  $\hat{K}$  on the horizontal axis and the proportional (deterministic part of the) change  $d\hat{K}/\hat{K}$  on the vertical one. The steady state  $\hat{K}^*$  to which the economy approaches without any jumps in technology is from (25) and (26)

$$\hat{K}^* = \left( \frac{1 - \tau_F}{1 + \tau_I} \frac{1 - \sigma}{b_1} \right)^{\frac{1}{1-\alpha}} L = \left( \frac{1 - \tau_F}{1 + \tau_I} \frac{\sigma}{\rho + \lambda [1 - (1 - s) \xi^{-\sigma}] + \frac{1 - \tau_F}{1 + \tau_I} \delta + \tau_W} \right)^{\frac{1}{1-\alpha}} L, \quad (28)$$

where we used (27) for the second equality.

We can now start our analysis as we do in deterministic models. Assume an initial capital stock  $\hat{K}_0$ . Agents invest part of their savings in R&D, the rest goes to capital accumulation. Assuming a certain length of time without jumps, i.e. without successful innovation, the economy grows due to more capital and converges to the steady state  $\hat{K}^*$ . As in the Solow model, growth is initially high and approaches zero. Once a jump occurs and  $q$  increases by 1, the capital stock of the economy increases by the size  $\kappa$  of the prototype from (3). If the capital stock  $K$  remained unchanged, capital per effective worker  $\hat{K}(\tau)$  from (17) would decrease by a discrete amount as the frontier technology increases by the discrete amount  $A$ . When we assume that the size of the new machine is sufficiently small relative to the

technological improvement,  $A^{-1}\xi < 1$  (which is the only empirically plausible assumption and which also follows from our assumption after (23)), the cyclical component  $\hat{K}(\tau)$  falls due to an innovation, i.e. the economy is thrown back towards the origin in fig. 1. With a lower capital stock per effective worker, investment in capital accumulation becomes more profitable as the marginal productivity of capital is higher. Growth rates jump to a higher level and approach zero again unless a new innovation takes place.

The discrete increases of labour productivity by  $A$  imply a step function in *vintage-specific* TFP. As investment takes place only in new vintages, each new vintage induces an investment boom. This is in contrast to the smooth increase in TFP for investment goods in balanced growth models. The implied evolution of GDP is shown in the right panel of fig. 1. Fluctuations are natural in a growing economy. The step function from one vintage to the next does not imply, however, that there are discrete jumps in TFP at the economy wide level. Looking at the aggregate technology (8) shows that capital can be aggregated to an index (9) which weights vintages according to their relative productivities. Prices of vintages fully reflect differences in productivity and *aggregate* TFP is therefore constant and equal to one.<sup>10</sup>

## 4 Measuring welfare and volatility

### 4.1 The value function

Our measure of welfare is the value function which, by definition, is  $V(t) \equiv \max_{\{c(\tau), i(\tau)\}} E_t \int_t^\infty e^{-\rho[\tau-t]} u(c(\tau)) d\tau$ . Pulling the expectations operator into the integral gives<sup>11</sup>

$$V(t) = \max_{\{c(\tau), i(\tau)\}} \int_t^\infty e^{-\rho[\tau-t]} E_t u(c(\tau)) d\tau. \quad (29)$$

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<sup>10</sup>There is an ongoing debate in the literature whether models of the natural volatility type are useful to think about business cycles. Andolfatto and MacDonald (1998) argue that diffusion waves of 7 new technologies can be identified since WWII in the US. According to this view, new technologies cause fluctuations at business cycles frequencies. A shortcoming of most existing papers is the lack of recessions (see, however, Francois and Lloyd-Ellis, 2003). Our model can in principle account either for high- or for low-frequency fluctuations. It remains an open empirical question whether new technologies alone cause business cycles or whether exogenous shocks (like oil-price shocks) are needed for convincing empirical "cycle accounting".

<sup>11</sup>The integral and the expectations operator can be exchanged when, under a technical condition, both the expected integral, i.e. our objective function (13), and the integral of the expected expression exist. The expected integral exists by assumption as otherwise the maximization problem of the household would be meaningless. The existence of the integral of the expected expression will be shown by computing it. The existence proof is therefore an ex-post proof. We are grateful to Ken Sennewald for discussions of this issue.

Obviously, the value of the optimal program depends on the evolution of expected instantaneous utility,  $E_t u(c(\tau))$ .

#### 4.1.1 Evolution of expected instantaneous utility

Let us now analyze how expected instantaneous utility,

$$m_1(\tau) \equiv E_t u(c(\tau)), \quad (30)$$

evolves. For notational simplicity, we denote

$$\Theta \equiv A^{1-\sigma}, \quad \Xi = \xi^{1-\sigma}. \quad (31)$$

Computing expected quantities as in (30) can be done in two ways. Either, a stochastic differential equation is expressed in its integral version, expectations operators are applied and the resulting deterministic differential equation is solved. Or, the stochastic differential equation is solved directly and then expectation operators are applied. The background for either approach is in Garcia and Griego (1994). We follow the first way here.

The evolution of  $u(c(\tau))$ , denoted by  $u(t)$  for simplicity, is described by the differential equation (app. C.1.1),

$$du(t) = \{b_0 \Theta^{q(t)} - b_1 u(t)\} dt - b_2 u(t) dq(t), \quad (32)$$

where  $b_0$  and  $b_1$  are as in (26) and (27) and

$$b_2 \equiv 1 - \Xi \quad (33)$$

can be understood as a measure of the “novelty” of a new technology. When  $A$  is high,  $b_2$  is high as well as a high degree of novelty increases  $b_2$  through high economic depreciation  $s$ , defined before (16). Note that we assume  $b_2 > 0$  which holds due to the plausibility assumption of  $\xi < 1$  made after (23). This differential equation shows that  $u(t)$  behaves similarly to  $\hat{K}$  illustrated in fig. 1. Starting from some  $u_0$ ,  $u(t)$  moves towards the current steady state  $b_0 \Theta^{q(t)}/b_1$  as long as no technology jump takes place, i.e. as long as  $dq = 0$ . When  $q$  jumps,  $u(t)$  reduces by a small amount as agents postpone consumption<sup>12</sup> and as a fraction of their wealth depreciates economically. The difference to  $\hat{K}$  consists in the behaviour of the current steady state. As  $u(t)$  is the *level* of utility and not its cyclical component or utility

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<sup>12</sup>This is due to  $\sigma = \alpha$  and the implied intertemporal elasticity of substitution. Under an alternative condition and closed form solution, consumption would not decrease (Wälde, 2005, footnote 20). The behavior of the utility level after a technological jump is not important for subsequent results.

per representative worker, the steady state moves to the right with each new technology. After an innovation and the subsequent reduction in  $u(t)$ , instantaneous utility approaches this new steady state until the next jump occurs - similar to GDP in fig. 1.

Given the stochastic differential equation (32) and forming expectations about  $u(\tau)$  for  $\tau > t$  leads to a deterministic ordinary differential equation in  $m_1(\tau)$ . Defining  $g$  as the growth rate and  $\beta$  as the convergence rate of expected utility  $m_1(\tau)$  (and keeping the difference to  $b_1$  in (27), the speed of convergence of  $\hat{K}$ , in mind),

$$g \equiv (\Theta - 1) \lambda, \quad (34)$$

$$\beta \equiv g + b_1 + b_2 \lambda = b_1 + \lambda[\Theta - \Xi], \quad (35)$$

respectively, we obtain (app. 7.1) an explicit expression for (30),

$$m_1(\tau) = e^{-(\beta-g)[\tau-t]} (u(t) - \mu) + e^{g[\tau-t]} \mu, \quad (36)$$

where

$$\mu \equiv \Theta^{q(t)} b_0 / \beta. \quad (37)$$

The second term of this equation,  $e^{g[\tau-t]} \mu$ , says that expected utility, starting in  $t$  where  $q(t)$  and  $K(t)$  and thereby  $u(t)$  are given as initial conditions, grows exponentially at the innovation rate  $g$ . From (34), the innovation rate is basically driven by the arrival rate  $\lambda$ . In the long run,  $g$  is the average growth rate of instantaneous utility. The first term says that  $u(t)$  converges to  $\mu$  at the convergence rate  $\beta$ . The term  $\mu$  is the expected value, today in  $t$ , of instantaneous utility in  $\tau \rightarrow \infty$ , when instantaneous utility is deterministically detrended. This follows immediately from rewriting (36) as  $e^{-g[\tau-t]} m_1(\tau) = e^{-\beta[\tau-t]} (u(t) - \mu) + \mu$ .<sup>13</sup> Somewhat imprecisely but nevertheless useful,  $\mu$  could be called “average instantaneous utility”.

Apart from showing growth of expected quantities in our setup, equation (36) illustrates the similarity of the evolution of expected quantities in this setup to the evolution of quantities in the Solow growth model. When we replace  $\mu$  by the Solow steady state utility level, the expected evolution here is identical to the certain evolution in Solow’s model (where  $g$

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<sup>13</sup>Obviously, detrending is possible in at least two ways here: The “stochastic detrending” in (17) looks at past realizations of  $q(t)$  and removes the stochastic trend  $A^{q(t)/\alpha}$  or  $A^{q(t)}$  of some stochastic trended variable  $X(t)$ . “Deterministic detrending” removes an expected growth trend by dividing expected expressions by its growth component  $e^{g[\tau-t]}$ . In either case, by definition, the resulting cyclical component has a finite constant long-run mean. Stochastic detrending also implies finite and constant higher long-run moments (app. 7.3), which, however, is not necessarily the case for deterministic detrending (app. 7.2).



and  $\beta$  would then stand for the growth and convergence rate in the Solow sense, respectively). In contrast, the role played by short-run convergence is ambiguous: while in the Solow model one usually assumes a capital stock that lies to the left of the steady state and convergence implies higher average growth rates between today and some future point in time  $\tau$ , the capital stock here (and the implied consumption and utility level) may lie left as well as right to the mean  $\mu$ . Convergence then implies higher or lower average growth rates than  $g$ , respectively.

#### 4.1.2 Computing the value function

We can now insert the expression for utility under optimal behaviour from (36) into (29) and get, after computing deterministic integrals (app. C.1.2),

$$V(t) = \frac{\mu}{\rho - g} - \frac{\mu - u(t)}{\rho - g + \beta}. \quad (38)$$

The derivation assumed  $\rho > g$  which makes the integral in (13) bounded.<sup>14</sup> While in deterministic models the growth rate of utility must not be larger than the time preference rate, in this stochastic model, the boundedness condition requires that the growth rate  $g$  of expected instantaneous utility must not exceed the time preference rate.

The value function can best be understood by going back to equation (36): The value to which the expected value of deterministically detrended utility converges is  $\mu$ . This value appears in (38) as  $\mu/(\rho - g)$ , i.e. the present value of utility that amounts to  $\mu$  today, grows at rate  $g$  and where the discount factor is  $\rho$ . In addition, welfare today depends on a convergence term. If utility today is lower than  $\mu$ , there will be convergence towards this long-run mean and utility will be lower compared to a situation where  $u(t)$  equals or exceeds  $\mu$ . However, the difference  $\mu - u(t)$  is not as important as  $\mu$  in the other term, as this effect is transitory only. Hence, the present value of the convergence process is computed subject to the convergence rate  $\beta$ .

Note that an identical expression for the value function would result in an analysis of the Solow model. The only difference would consist in the meaning of  $\mu$ . While here,  $\mu$  stands for “average instantaneous utility”, it would stand for steady state utility in Solow’s model.

Summarizing, the value of the optimal program  $V(t)$  basically depends on four crucial determinants: “average instantaneous utility”  $\mu$ , utility today  $u(t)$ , the innovation rate  $g$  and the convergence coefficient  $\beta$ . Studying welfare effects of taxation can therefore be broken down into effects on these four elements that determine the value function.

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<sup>14</sup>Hence, the integral of the expected expression exists. See footnote 11.

## 4.2 The cyclical component

While the measure for welfare was straightforward, there is an almost infinite number of possible measures of volatility. The empirically oriented literature provides two examples: The variance of growth rates (e.g. of GDP) and the variance of cyclical components. App. 7.2 analyses the variance of the growth rate of instantaneous utility  $u$  in detail. It turns out that the resulting expression and therefore variances of all other time-series like e.g. GDP, do not lend themselves to a straightforward analysis. This is due to two facts. First, growth rates for long time horizons, i.e. for  $\tau \rightarrow \infty$ , do approach a constant mean but do not have finite variance or finite higher moments. Second, while annual growth rates have finite moments, they are extremely complex (cf. eq. (51)) and a comparative static analysis is close to intractable. We therefore base our measure of volatility on cyclical components.

### 4.2.1 The evolution of the cyclical component

Cyclical components of time series can be defined and therefore computed in many ways and the literature offers a large number of filters. None of these filters, given their computational complexity, would allow us to derive cyclical components that would yield an explicit analytical expression for volatility. We therefore use a very simple filter, the Solow-type detrending rule used in (17), to compute our cyclical components. Usual filters, think of the Hodrick-Prescott filter, detrend by removing a smooth trend from a time series. Our filter captures the trend by a step function  $A^{q(t)}$ , caused by the discrete increases of  $q(t)$ . In spirit, however, these filters are very close as both remove past realizations of growth processes to obtain the cyclical components.

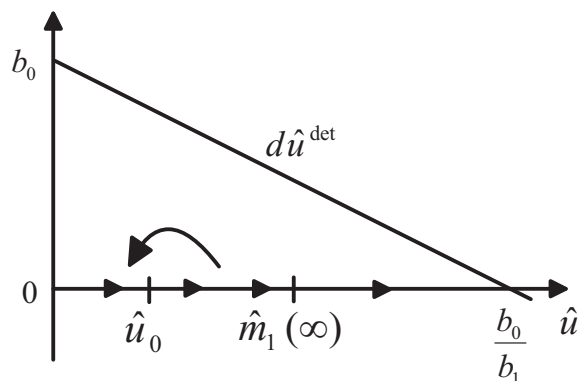


Figure 2: Detrended instantaneous utility

In order to understand the detrending method proposed here, we look at the evolution of detrended utility.<sup>15</sup> We define detrended individual utility, in analogy to (14), as the component of utility that stems from the cyclical component of consumption in (17),

$$\hat{u} = \frac{(\hat{C}/L)^{1-\sigma}}{1-\sigma}. \quad (39)$$

With

$$\hat{b}_2 \equiv 1 - \Xi/\Theta, \quad (40)$$

detrended utility follows (app. 7.3)

$$d\hat{u}(t) = \{b_0 - b_1\hat{u}(t)\} dt - \hat{b}_2\hat{u}(t) dq(t). \quad (41)$$

This law of motion is basically identical to (32), only that the  $\Theta^{q(t)}$  term is missing and  $\hat{b}_2$  slightly differs from  $b_2$ . Again, we can gain an intuitive understanding by plotting in fig. 2 the deterministic part  $(b_0 - b_1\hat{u}(t))$  with  $\hat{u}(t)$  on the horizontal axis.

Obviously, the cyclical component of utility has a range between 0 and  $b_0/b_1$ , provided that  $\hat{u}_0$  lies within this range. Starting from  $\hat{u}_0$  and as long as no innovation takes place, the cyclical component approaches its upper bound. Each innovation reduces  $\hat{u}(t)$  to  $(1 - \hat{b}_2)\hat{u}(t)$ , i.e.  $\Xi/\Theta$  percent of its level before the innovation. As the reduction is proportional,  $\hat{u}(t)$  is always positive.

#### 4.2.2 The coefficient of variation

Exploiting again the methods in Garcia and Griego (1994), we can compute moments of this cyclical component. This follows similar step as above for (36). In fact, denoting the  $i$ th moment, in analogy to (30), by

$$\hat{m}_i(\tau) \equiv E_t \hat{u}(\tau)^i, \quad (42)$$

the first and second moment are given in the long run by (app. 7.3.2)<sup>16</sup>

$$\hat{m}_1(\infty) = \frac{b_0}{b_1 + \lambda \hat{b}_2}, \quad (43)$$

$$\hat{m}_2(\infty) = \frac{2b_0}{2b_1 + \lambda [1 - (1 - \hat{b}_2)^2]} \hat{m}_1(\infty). \quad (44)$$

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<sup>15</sup>In Lucas-type approaches, the measure of volatility is based on consumption. For analytical tractability, we work with detrended utility. There are approximation rules which allow to compute e.g. the coefficient of variation ( $cv$ ) of consumption once the  $cv$  of utility (a monotone transformation of consumption) is known.

<sup>16</sup>The structure of the moments is remarkable as it shows that the distribution of  $\hat{u}$  exists, is unique and represents a generalization of the  $\beta$ -distribution. We are grateful to Christian Kleiber for pointing this out to us. For more discussion see app. 7.3.2.

Using these moments, computing the variance would be straightforward. As a measure of volatility, the variance seems less suitable in our context, however, as it is scale dependent. We therefore prefer the coefficient of variation ( $cv$ ). Given that the variance of a random variable is the difference between its second moment and the square of its mean, we obtain

$$cv^2 \equiv \frac{\lim_{\tau \rightarrow \infty} \text{var}_t \hat{u}(\tau)}{(\lim_{\tau \rightarrow \infty} E_t \hat{u}(\tau))^2} = \frac{\hat{m}_2(\infty)}{(\hat{m}_1(\infty))^2} - 1. \quad (45)$$

When computing the second moment in all generality, an expression similarly complex as for the variance of the growth rate, presented in the appendix in (51) appears. When we focus on the long run where the convergence of the initial value  $\hat{u}_0$  to  $\hat{m}_1(\infty)$  in (41) is ignored, however, this measure simplifies. This would be the case for the variance of growth rates as well, see (52). Studying the long run with this measure of volatility is not at all as problematic as using growth rates, however. In the latter case, we analyze the variance of multi-annual growth rates. Those could never be observed. In the former case, looking at the long run simply means studying the volatility of some stationary long-run distribution. This corresponds to studying the variance of the cyclical component of a time series that is very long. This being said, our  $cv$  is (app. C.3.2)

$$cv^2 = \frac{\hat{b}_2^2}{2b_1/\lambda + 1 - (1 - \hat{b}_2)^2}. \quad (46)$$

To obtain a feeling for this measure, we go back to fig. 2. The first moment  $\hat{m}_1(\infty)$  lies between 0 and the steady state  $b_0/b_1$ . This is intuitively clear, given the permanent convergence towards  $b_0/b_1$  and the occasional being thrown back. As the process  $\hat{u}(t)$  is completely described by (41), given an arrival rate  $\lambda$ , only the parameters of this process,  $b_0$ ,  $b_1$ ,  $\hat{b}_2$  and  $\lambda$ , can show up in its moments. A larger  $b_0$  and a smaller  $b_1$  shifts the mean  $\hat{m}_1(\infty)$  to the right; this is clear from fig. 2 as a larger  $b_0$  and a smaller  $b_1$  shift the  $d\hat{u}$  line to the right. When  $\hat{b}_2$  or  $\lambda$  increase, the mean shifts to the left as either jumps to the left are larger or more frequent.

The second moment has the same properties as the first moment  $\hat{m}_1(\infty)$  with respect to  $b_0$ ,  $b_1$  and  $\lambda$ , as can be directly seen in (44). As the term  $1 - (1 - \hat{b}_2)^2$  increases in  $\hat{b}_2$ , it also behaves as  $\hat{m}_1(\infty)$  with respect to  $\hat{b}_2$ , i.e. it decreases in  $\hat{b}_2$ . Simply speaking, a larger range and more frequent jumps increase the second moment, a measure of dispersion.

Computing the  $cv$  then shows that it is independent of  $b_0$ . This is not surprising as  $b_0$  is a scaling parameter and the  $cv$  is by construction scale independent. This can intuitively also be understood from fig. 2 where the effect of  $b_0$  on the cyclical component could be removed by scaling both axes with  $1/b_0$ . The effect of other parameters will be discussed below.

### 4.2.3 Random walks and stationary cyclical components

Following the work of Nelson and Plosser (1982), many economists now argue that macroeconomic time series exhibit difference-stationarity rather than trend-stationarity. This implies that theoretical models should predict difference stationary time paths as well, i.e. trends and cyclical components should both be stochastic as opposed to models where only cyclical components are stochastic and the trend is deterministic.

We now show that our model exhibits indeed a stochastic trend and stochastic stationary cyclical components.<sup>17</sup> We can write (17) as  $\ln K(t) = q(t) (\ln A) / \alpha + \ln \hat{K}(t)$ , i.e. we split our time series  $\ln K(t)$  into a trend component  $q(t) (\ln A) / \alpha$  and into a stationary cyclical component  $\ln \hat{K}(t)$ , in the sense of Beveridge and Nelson (1981). Both the trend component and the stationary component are stochastic. Even though we are in continuous time, we can easily relate our trend component to a discrete time random walk as we can describe it by the pure random walk with drift:  $q(t) = q(t-1) + \lambda + \varepsilon(t)$ , where  $\varepsilon(t) \sim (0, \lambda)$ .<sup>18</sup> Hence, our trend component has a unit root and our cyclical component  $\hat{K}(t)$  is stationary as just shown for  $\hat{u}$ .

## 5 Volatility, welfare and taxation

Given our measures of welfare and volatility derived in the last section, we can now ask how taxation affects these quantities.

### 5.1 Volatility and taxation

#### 5.1.1 The volatility channels

Our central measure of volatility in (46) is affected through three channels: The speed of convergence  $b_1$ , the altitude of the slump  $\hat{b}_2$  and the arrival rate  $\lambda$ . The interpretation of

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<sup>17</sup>Other models of endogenous fluctuations and growth, all cited in the introduction, are of a deterministic nature. The only exception is Bental and Peled (1996) who were the first to study endogenous fluctuations and growth. Unfortunately, their model is fairly complex which makes an explicit analysis of stochastic properties of trends and cycles a very hard task.

<sup>18</sup>The fact that the expectation and variance of  $q(t) - q(t-1)$  are both equal to  $\lambda$  results from the distributional properties of a Poisson process. If the increment of the trend term was not constant, i.e. if e.g.  $A$  was vintage dependent and stochastic, the expectation and variance would differ. This would be an interesting extension for future work and should help in empirical applications. See Sennewald and Wälde (2006, sect. 3.4.2) on how to disentangle the expected value and variance of a Poisson driven process.

these parameters is based on (41) but other interpretations are possible. When we plot an arbitrary realization of our cyclical components in fig. 3, this becomes more transparent.

The range of our cyclical components is  $]0, b_0/b_1[$ . The upper limit corresponds to the steady state  $\hat{K}^*$  for the cyclical component of capital in fig. 1. Hence,  $b_1$  is at the same time a measure of the range of the cyclical component (cf. fig. 2, remembering that  $b_0$  is only a scaling parameter) and thereby of its amplitude. The arrival rate  $\lambda$  also measures the expected number of jumps or (the inverse of) the expected length of a cycle. The simple reason why volatility depends on taxation is therefore the same reason why the steady state capital stock (28) (i.e. the speed of convergence), the novelty of a new technology or the arrival rate depend on taxation.

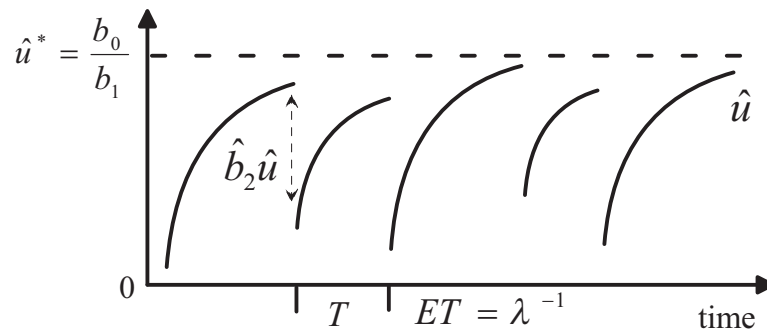


Figure 3: The cyclical components and their determinants  $b_1$ ,  $\hat{b}_2$  and  $\lambda$

When we want to understand the effects of taxation, we can restrict attention to  $b_1$  and  $\lambda$  as  $\hat{b}_2$  is independent of taxation. The independence of  $\hat{b}_2$  (and of  $b_2$ ) from taxes follows from their definitions in (33) and (40) and the fact that  $\xi$  from (23), with  $s$  from (16) and  $\kappa_0$  from (4), is independent of tax rates. Economically, this independence of  $\hat{b}_2$  follows from the fact that dividend payments  $\kappa$  are not taxed and that economic depreciation  $s$  does not imply tax-exemption as does physical depreciation  $\delta$ .

The tax effects on the arrival rate  $\lambda$  are straightforward from looking at (22) and are summarized in table 1. As the growth rate  $g$  of expected utility has  $\lambda$  as its only tax-dependent determinant, it has the same qualitative properties and is also included in the table. The composite parameter  $b_1$  in (27) depends on taxes both directly and indirectly through the arrival rate. When we insert (22) into (27), we obtain unambiguous results,

except for  $\tau_I$  (app. D.1.1).

	$\tau_F$	$\tau_C$	$\tau_R$	$\tau_I$	$\tau_W$
$b_1$	-	0	-	+ <sup>(1)</sup>	+
$b_2, \hat{b}_2$	0	0	0	0	0
$g, \lambda$	0	0	-	+	0
volatility	+	0	-	+	-
welfare	-	-	$\pm$	$\pm$	-

<sup>(1)</sup> for high  $\tau_I$  or  $\tau_F$  and low  $\delta$

Table 1: Taxation effects on composite parameters, the arrival rate, volatility and welfare

### 5.1.2 Comparative statics

Let us now combine the effects of these three channels on volatility. As we have only two tax-dependent channels,  $b_1$  and  $\lambda$ , taxation can affect volatility by either changing  $\lambda$  or  $b_1$  (without the  $\lambda$  in  $b_1$ ), or both. Clearly, when a tax has no effect on  $b_1$  and  $\lambda$ , the  $cv$  is not affected by this tax either. This is the case for taxation of consumption.

When taxing wealth, the arrival rate and the “slump parameter”  $\hat{b}_2$  are not affected, while the (inverse) range parameter  $b_1$  increases and, as a consequence, volatility goes down. Economically speaking, a tax on wealth decreases the households’ return  $r^*$  in (16) on savings and thereby implies a lower steady-state cyclical capital stock and utility level  $\hat{u}$ . Holding constant the length of a cycle but “squeezing” the cyclical components in fig. 3, the relative dispersion must be lower.

An increase in the income tax on capital and labour reduces the range parameter  $b_1$  but not the arrival rate  $\lambda$ . As a consequence, volatility unambiguously increases in this tax. How can this result be understood? The speed of convergence  $b_1$  in (27) reduces for two reasons: (i) only *net* investment is taxed (as discussed before (15)), i.e. a higher tax on capital increases the positive effect of the refunding policy and reduces the impact of the depreciation rate  $\delta$  as visible in (27). A lower (effective) depreciation rate increases incentives for capital accumulation and the steady-state capital stock increases. (ii) Due to our  $\sigma = \alpha$  restriction, direct effects of joint changes in capital and labour taxation just cancel out and only this indirect refunding effect is left over. Clearly, this second effect would not survive for  $\alpha \neq \sigma$  and should potentially overcompensate the first effect. Hence, currently, the effect of income taxation is the opposite of wealth taxation but should go in similar directions for more general cases.

When analyzing R&D and investment taxes  $\tau_R$  and  $\tau_I$ , results are at first sight less clear-cut as these taxes affect the arrival rate which affects the  $cv$  directly positively and indirectly negatively through  $\lambda$ . Computing the derivatives, however, we get unambiguous results as presented in table 1 above (app. D.1.2). The analytics for  $\tau_R$  say in words: a higher tax on research depresses the arrival rate. When the arrival rate falls, the ratio  $b_1/\lambda$  increases and the  $cv$  falls. Intuitively, a higher  $\tau_R$  makes investment in research less profitable and the arrival rate  $\lambda$  falls. Less frequent jumps in technology imply a lower volatility. A lower  $\lambda$  also decreases  $b_1$  which by fig. 3 implies a larger range  $b_0/b_1$  and higher volatility. This is because  $b_1$  represents physical depreciation but also consumption and expenditure for research. A lower  $\lambda$  therefore implies ceteris paribus a lower  $b_1$ . This second indirect effect, however, is not large enough such that the direct volatility decreasing effect of higher taxes on research dominates.

Given the explanations for the previous finding, understanding the result for  $\tau_I$  is also easy: a higher tax on investment increases the arrival rate which again has a direct and an indirect effect on volatility. The direct effect via the frequency of jumps overcompensates the indirect effect on the range and volatility increases. The additional effect of taxation on investment via the depreciation rate  $\delta$  makes the range increasing effect of a higher  $\lambda$  less strong such that the indirect effect is even weaker than under a change of  $\tau_R$ . Consequently, volatility falls more when  $\tau_I$  increases as when  $\tau_R$  falls.

If growth and volatility were exogenous, i.e. if there was an exogenous arrival rate  $\lambda$  without any resources  $R$  being used for R&D, the model would from its basic structure resemble a simple RBC model. Any activity takes place under perfect competition and labour productivity improves by discrete amounts at random points in time. Volatility would still be affected by taxation as the arrival rate is only one out of three channels in our measure of volatility (46).<sup>19</sup> As McConnell and Perez-Quiros (2000) argue, however, there was a break in volatility in the US in 1984 *without* a break in the trend of GDP. Hence, we can meaningfully investigate whether taxes can explain a break in volatility without affecting the growth rate only with an endogenous arrival rate. As our results in table 1 show, this model predicts indeed that a falling income tax  $\tau_F$  and a rising wealth tax  $\tau_W$  reduce volatility without affecting trend growth.

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<sup>19</sup>The amount of volatility would therefore remain endogenous even in this exogenous shock economy. Volatility could, however, no longer be called “natural” as its source is exogenously imposed on the economy.



## 5.2 Volatility and welfare

### 5.2.1 The welfare channels and comparative statics

When we look at our measure of welfare (38), it is affected by taxation through four quantities, average instantaneous utility  $\mu$ , current utility  $u(c(t))$ , and  $g$  and  $\beta$ , the growth and convergence rate. These four quantities in turn depend on four channels,  $b_0$ ,  $b_1$ ,  $\lambda$  and  $\Psi$ . We could now, following the approach from our measure of volatility, analyze the effects of taxation on these channels first and then combine the results and derive conclusions for welfare. As this does not yield additional insight, we directly link welfare to taxation by the following

**Theorem 2 (Taxation and welfare)** *A tax reduces welfare (38) when the permanent component of welfare  $\mu/(\rho - g)$  falls faster or increases less fast than the transition component  $(\mu - u(t))/(\rho - g + \beta)$ . Computing the derivatives, we get*

$$\begin{aligned} & \frac{\left(\frac{\partial b_0}{\partial \tau_i} \frac{1}{b_0} - \frac{\partial \beta}{\partial \tau_i} \frac{1}{\beta}\right) (\rho - g) + \frac{\partial \lambda}{\partial \tau_i} (\Theta - 1)}{(\rho - g)^2} \\ & < \frac{\left(\frac{\partial b_0}{\partial \tau_i} \frac{1}{b_0} - \frac{\partial \beta}{\partial \tau_i} \frac{1}{\beta} - \frac{u(t)}{\mu} \frac{1-\sigma}{\Psi} \frac{\partial \Psi}{\partial \tau_i}\right) (\rho - g + \beta) - \left(\frac{\partial b_1}{\partial \tau_i} + \frac{\partial \lambda}{\partial \tau_i} b_2\right) \left(1 - \frac{u(t)}{\mu}\right)}{(\rho - g + \beta)^2}. \end{aligned}$$

**Proof.** app. D.2 ■

The left-hand side is the derivative of the permanent component of welfare, the right-hand side is the derivative of the transition component (where both sides are divided by  $\mu$ ). Both derivatives can be both negative or positive.

Going through these derivatives for individual taxes shows that (compare table 1 and app. D.2) taxes on factor income, consumption and wealth have unambiguous negative effects on welfare while taxes on investment and R&D can increase welfare. Taxes  $\tau_F$ ,  $\tau_C$  and  $\tau_W$  decrease welfare as resources are taken away from households and  $G$  has in the model no productivity- or utility-enhancing effect. The potentially welfare increasing effect of  $\tau_I$  and  $\tau_R$  can best be understood when looking at the first order condition for investment in research: in our decentralized setup, the first order condition  $(1 + \tau_R) V_a(a(t), q) = \lambda V_{\tilde{a}}(\tilde{a}, q + 1) \frac{(1 + \tau_I) \kappa}{R}$  from (A.17) shows that individuals invest in R&D because of dividend payments  $\kappa^*$ , a higher wealth level  $\tilde{a}$  after successful R&D and the better technology  $q + 1$ . Optimal investment in a planner economy, where the planner maximizes the Bellman equation (A.15) with respect to  $R$  rather than  $i$  and where  $\Sigma a(t)$  stands for wealth in the economy as a whole, would satisfy  $V_{\Sigma a}(\Sigma a(t), q) = \frac{\partial \lambda}{\partial R} [V(\Sigma \tilde{a}, q + 1) - V(\Sigma a, q)]$ . Incentives

to do research therefore results from  $\partial\lambda/\partial R$ , the effect of more resources on the probability  $\lambda$  to find a new technology, and the difference in well-being between a situation with more wealth and a better technology,  $V(\Sigma\tilde{a}, q + 1)$ , and today,  $V(\Sigma a, q)$ . While there are certainly various opposing effects, externalities are strongest for this trade-off between capital accumulation and R&D. It is therefore not surprising, that taxes  $\tau_I$  and  $\tau_R$  which are directly affecting this first order condition are best suited to potentially internalize externalities.

As the first order conditions for consumption is identical for the planner and the representative household, there are no externalities or imperfections present in the model apart from those visible in the difference between the first order conditions for R&D. Put differently, if the arrival rate equalled  $\lambda$  exogenously without any resources  $R$  being allocated to R&D, the decentralized economy would be efficient. This RBC-type version of our model would then predict that fluctuations allow for an optimal adjustment by individuals to exogenous disturbances. If one believes, however, that the process of finding and developing new technologies implies certain externalities (and that new technologies at least partially induce fluctuations), factor allocations in an economy “growing through cycles”, to use Matsuyama’s (1999) words, are inefficient.

### 5.2.2 The tax-link between volatility and welfare

Given the inefficiency of fluctuations, should taxes be used to stabilize the economy? In the literature, more volatility is usually associated with lower welfare: In perfect-market models (Lucas, 1987 or, more recently, Barlevy, 2004), exogenous volatility implies fluctuations of consumption and the curvature of the utility function implies lower welfare than in an economy without fluctuations but identical average growth. Galí et al. (2003) focus on inefficiencies and argue - due to the inefficiency of the steady state and the larger welfare losses in recessions than welfare gains in booms - that fluctuations on average cause welfare losses.

This is not necessarily the case when fluctuations are endogenous: While the curvature of the utility function à la Lucas and the asymmetry as in Galí (and others) is welfare-reducing in our setup as well, volatility is only the result of the more fundamental factor allocation in an economy. Asking whether volatility is welfare reducing and by how much is therefore meaningful only if one believes that the sources of fluctuations are exogenous to an economy (which, in the real world, they are - to a certain extent). The welfare effects of endogenous fluctuations, however, can only be understood when understanding the welfare properties of the underlying factor allocation that causes these fluctuations. When this is done, it becomes

clear that more or less fluctuations can be associated (and are not causal as in the exogenous fluctuation case) with higher or lower welfare. Tax policy should therefore not be used in all cases to stabilize the economy.

This association between welfare and volatility, illustrated for taxes with unambiguous welfare effects, is depicted in the following figure.

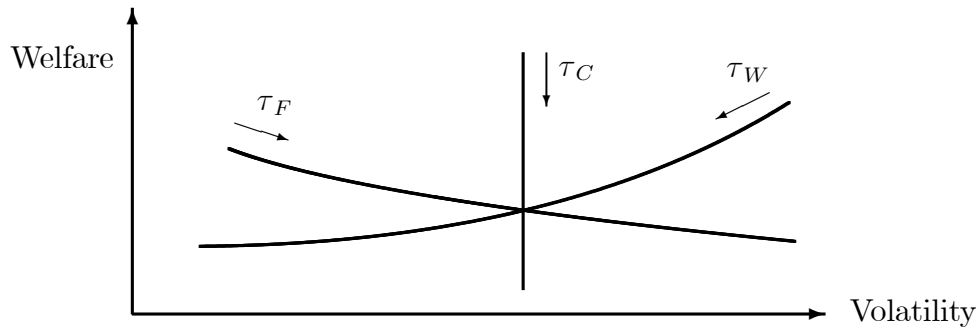


Figure 5: The effect of taxation on welfare and volatility

Arrows indicate in which direction to move on the line when the tax is increased. Fig. 5 shows that there is no association between volatility and welfare in general. It all depends on the source of the change in volatility. While certain taxes increase volatility and reduce welfare ( $\tau_F$ ), others reduce volatility but still decrease welfare ( $\tau_W$ ). Lowering the tax on wealth increases welfare as fewer resources are taken away from the economy as argued above. At the same time, volatility increases as the steady state capital stock (28) increases. Hence, despite the curvature of the utility function and the asymmetric effect of volatility on efficiency, more volatility implies higher welfare.

## 6 Conclusion

Growth rates above and below long-run trends are a common feature of all real-world economies. The present paper used a model that perceives endogenous fluctuations as a natural phenomenon of all endogenously growing economies by stipulating that new technologies increase labour productivity in a discrete way. Agents in this setup are not solely responding to shocks but rather are the source of shocks, i.e. jumps in technologies, due to their financing of innovation and growth. This framework was used to analyze the effects of taxation on volatility and the associated welfare effects. The motivation for this is provided by the sharp decrease of volatility in the US around 1983 and an almost simultaneous strong tax reform.

We used the coefficient of variation of the cyclical component of a typical time series as our measure of volatility. It was shown that this measure is most tractable from a theoretical perspective and that three economically meaningful channels affect this measure: potential range of cyclical components, slumps and frequency of new technologies. Taxes affect these channels in various ways which allows, *inter alia*, to understand a change in volatility without requiring a simultaneous change in the growth rate of the economy.

Welfare effects associated with changes in volatility can be manifold. In a special case of our model where the source of long-run growth and short-run fluctuations is exogenous, factor allocation is efficient and volatility does not signal the need for stabilization. With endogenous growth and fluctuations, however, inefficiencies enter the economy and fluctuations hint at the possibility of welfare-increasing policy measures, even though all production and R&D activities were modelled to take place under perfect competition.

Stabilization is not necessarily welfare increasing, however: Lower volatility can imply higher or lower welfare, depending on whether the tax change reducing volatility implies higher or lower welfare. Analyzing the link between volatility and welfare should therefore not be restricted to the usual mono-causal link from an exogenous source of volatility and an endogenous welfare reaction but expanded to exogenous change in fiscal policy (or other exogenous changes) and how natural volatility and welfare react to this.

An important extension of the present analysis (and other papers in the literature on endogenous fluctuations and growth) would combine endogenous and exogenous sources of fluctuations. It appears reasonable to start an analysis of fluctuations of any real world economy by allowing for both endogenous jumps of and exogenous shocks to the technology as well as nominal sources of fluctuations. Labour market participation decisions and unemployment should also be included in future work. The implications of our analysis for the growth and volatility debate could also be worked out more precisely. With endogenous volatility, taxes (or other policy parameters) affect both long-run growth and volatility. As in our welfare argument, the causal link from volatility to growth becomes a correlation. The implied endogeneity of volatility in regression analyses could be tested. Finally, quantitative implications of this approach can be explored.

## 7 Appendix

This appendix contains derivations that are interesting from a theoretical perspective beyond this specific paper. Section 7.1 derives the evolution of expected instantaneous utility. It uses

methods that were developed in the applied mathematical literature, e.g. Garcia and Griego (1994). These methods are potentially useful also in other areas where Poisson processes are used (e.g. all search and matching models in monetary or labour economics). Section 7.2 computes an explicit expression for the variance of the growth rate. Again, various methods are borrowed from Garcia and Griego (1994). Finally, section 7.3 computes the moments of our basic random variable. This forms the basis for our measure of volatility. Interestingly, we obtain a generalized  $\beta$ -distribution from this analysis.

Further derivations are included in the Referees' appendix which is available upon request.

## 7.1 Evolution of expected instantaneous utility

This section computes the expected value of instantaneous utility, conditional on the current state in  $t$ , given by  $q(t)$  and  $K(t)$ . The results provide information about expected growth but are especially needed for computing the value function.

### 7.1.1 A lemma for $E(c^{kN_\tau})$

We first compute some simple expectations that are used later.

**Lemma 3** *Assume that we are in  $t$  and form expectations about future arrivals of the Poisson process. The expected value of  $c^{kq(\tau)}$ , conditional on  $t$  where  $q(t)$  is known, is*

$$E_t(c^{kq(\tau)}) = c^{kq(t)} e^{(c^k - 1)\lambda(\tau - t)}, \quad \tau > t, \quad c, k = \text{const.}$$

Note that for integer  $k$ , these are the raw moments of  $c^{q(\tau)}$ .

**Proof.** We can trivially rewrite  $c^{kq(\tau)} = c^{kq(t)} c^{k[q(\tau) - q(t)]}$ . At time  $t$ , we know the realization of  $q(t)$  and therefore  $E_t c^{kq(\tau)} = c^{kq(t)} E_t c^{k[q(\tau) - q(t)]}$ . Computing this expectation requires the probability that a Poisson process jumps  $n$  times between  $t$  and  $\tau$ . Formally,

$$\begin{aligned} E_t c^{k[q(\tau) - q(t)]} &= \sum_{n=0}^{\infty} c^{kn} \frac{e^{-\lambda(\tau - t)} (\lambda(\tau - t))^n}{n!} = \sum_{n=0}^{\infty} \frac{e^{-\lambda(\tau - t)} (c^k \lambda(\tau - t))^n}{n!} \\ &= e^{(c^k - 1)\lambda(\tau - t)} \sum_{n=0}^{\infty} \frac{e^{-\lambda(\tau - t) - (c^k - 1)\lambda(\tau - t)} (c^k \lambda(\tau - t))^n}{n!} \\ &= e^{(c^k - 1)\lambda(\tau - t)} \sum_{n=0}^{\infty} \frac{e^{-c^k \lambda(\tau - t)} (c^k \lambda(\tau - t))^n}{n!} = e^{(c^k - 1)\lambda(\tau - t)}, \end{aligned}$$

where  $\frac{e^{-\lambda\tau} (\lambda\tau)^n}{n!}$  is the probability of  $q(\tau) = n$  and  $\sum_{n=0}^{\infty} \frac{e^{-c^k \lambda(\tau - t)} (c^k \lambda(\tau - t))^n}{n!} = 1$  denotes the summation of the probability function over the whole support of the Poisson distribution which was used in the last step. ■

**Lemma 4** *Assume that we are in  $t$  and form expectations about future arrivals of the Poisson process. Then the number of expected arrivals in the time interval  $[\tau, s]$  equals the number of expected arrivals in an unknown time interval of the length  $\tau - s$  and therefore*

$$E_t(c^{k[q(\tau)-q(s)]}) = E(c^{k[q(\tau)-q(s)]}) = e^{(c^k-1)\lambda(\tau-s)}, \quad \tau > s > t, \quad c, k = \text{const.}$$

**Proof.** This proof is in appendix C.1.1, it simply applies lemma 3. ■

### 7.1.2 Expected instantaneous utility

We will use in what follows the martingale property of various expressions. These expressions are identical to or special cases of  $\int_t^\tau f(q(s), s) dq(s) - \lambda \int_t^\tau f(q(s), s) ds$ , of which Garcia and Griego (1994, theorem 5.3) have shown that it is a martingale indeed, i.e.

$$E_t \left[ \int_t^\tau f(q(s), s) dq(s) - \lambda \int_t^\tau f(q(s), s) ds \right] = 0, \quad (47)$$

where  $\lambda$  is the (constant) arrival rate of  $q(s)$ .

The integral version of (32) for  $\tau > t$  is  $u(\tau) = u(t) + \int_t^\tau b_0 \Theta^{q(s)} - b_1 u(s) ds - \int_t^\tau b_2 u(s) dq(s)$ . Applying (conditional) expectation operators gives  $E_t u(\tau) = u(t) + E_t \int_t^\tau b_0 \Theta^{q(s)} ds - E_t \int_t^\tau b_1 u(s) ds - E_t \int_t^\tau b_2 u(s) dq(s)$ . When we pull expectations into the integral (as in eq. (29)), use lemma 3 and the martingale result (47), we get  $E_t u(\tau) = u(t) + \Theta^{q(t)} \int_t^\tau b_0 e^{(\Theta-1)\lambda[s-t]} ds - \int_t^\tau b_1 E_t u(s) ds - \lambda \int_t^\tau b_2 E_t u(s) ds$ . With  $m_1(\tau) \equiv E_t u(\tau)$  from (30), we get  $m_1(\tau) = u(t) + \int_t^\tau \Theta^{q(t)} b_0 e^{(\Theta-1)\lambda[s-t]} ds - \int_t^\tau b_1 m_1(s) ds - \int_t^\tau b_2 \lambda m_1(s) ds$  and differentiating with respect to time  $\tau$  gives

$$\dot{m}_1(\tau) = \Theta^{q(t)} b_0 e^{(\Theta-1)\lambda[\tau-t]} - (b_1 + \lambda b_2) m_1(\tau). \quad (48)$$

The solution of this deterministic linear differential equation is

$$\begin{aligned} m_1(\tau) &= m_1(t) e^{-(b_1 + \lambda b_2)(\tau-t)} + \int_t^\tau \Theta^{q(t)} b_0 e^{(\Theta-1)\lambda[s-t]} e^{-(b_1 + \lambda b_2)(\tau-s-\tau+t)} e^{-(b_1 + \lambda b_2)(\tau-t)} ds \\ &= e^{-(b_1 + \lambda b_2)(\tau-t)} \left( m_1(t) + \Theta^{q(t)} b_0 \int_t^\tau e^{(b_1 + \lambda b_2 + (\Theta-1)\lambda)(s-t)} ds \right) \\ &= e^{-(\beta-g)(\tau-t)} \left( u(t) + \Theta^{q(t)} b_0 \frac{e^{\beta[\tau-t]} - 1}{\beta} \right), \end{aligned}$$

where the last line used  $m_1(t) = E_t u(t) = u(t)$  and (35) for  $\beta$ . Rearranging gives (36) in the text.

## 7.2 The variance of the growth rate

This section derives an alternative expression for volatility, the variance of the growth rate. This measure is more common in empirical work (e.g. Ramey and Ramey, 1995 or McConnell

and Perez-Quiros, 2000) than the variance of cyclical components, which in turn is used more intensively in the RBC literature.

It is not immediately clear, however, how this variance should be computed. Is it the variance of some long-run stationary distribution,  $\lim_{\tau \rightarrow \infty} \text{var}_t [g_{\tau,t}]$ , is it the variance of some “annual” growth rate of some long-run distribution,  $\lim_{\tau \rightarrow \infty} \text{var}_t [g_{\tau+1,\tau}]$ , or is it the variance of the next “period” in this model,  $\text{var}_t [g_{t+1,t}]$ ? In a way, the choice of measure of variance is arbitrary. We therefore choose the one that comes closest to the estimation of the variance of observed growth rates. The counterpart to an observed annual growth rate for a “year”  $t$  in our model is  $g_{t+1,t}$ . Taking many drawings, there is a set of annual growth rates  $\{g_{t+1,t}\}$  for which the variance can be estimated. Noting that annual growth rates are computed given the knowledge on  $t$ , the analytical expression corresponding to this is the  $t$ -contingent variance of  $g_{t+1,t}$ , i.e.  $\text{var}_t [g_{t+1,t}]$ .

Now, we can take advantage of the following straightforward relationship: The  $t$ -contingent variance of the growth rate of some random variable is the same as the  $t$ -contingent variance of the random variable divided by some constant. In our case,

$$\text{var}_t \left[ \frac{u(\tau) - u(t)}{u(t)} \right] = \frac{\text{var}_t u(\tau)}{u(t)^2}. \quad (49)$$

While this is trivial in a sense, it has the huge advantage that we can just compute the second moment of  $u(\tau)$  and thereby obtain the theoretical counterpart of the variance of observed growth rates.

The variance of  $u(\tau)$  is computed by first computing its second moment. To this end, the evolution of squared utility needs to be understood. It follows (app. C.2.2)

$$du(t)^2 = 2 \{b_0 \Theta^{q(t)} u(t) - b_1 u(t)^2\} dt - \{1 - (1 - b_2)^2\} u(t)^2 dq.$$

Comparing it to (32) shows that the main difference, apart from the square term  $u^2$  instead of  $u$ , is the interaction  $\Theta^q u$  between  $\Theta^q$  and  $u$ . When forming expectations, we therefore have to compute the expected interaction term, i.e. look at  $\psi(s) \equiv E_t \Theta^{q(s)} u(s)$ . After “some steps” (filling 6 pages in app. C.2), denoting

$$\begin{aligned} g_\psi &\equiv (\Theta^2 - 1) \lambda > 0, \\ \beta_\psi &\equiv g_\psi + b_1 + (1 - \Theta [1 - b_2]) \lambda = b_1 + (\Theta^2 - \Theta \Xi) \lambda > 0, \\ \beta_2 &\equiv g_\psi + 2b_1 + (1 - (1 - b_2)^2) \lambda, \end{aligned} \quad (50)$$

the variance from (A.49) is

$$\begin{aligned}
var_t(u(\tau)) &= \mu^2 \left[ e^{-(\beta_2 - g_\psi)(\tau - t)} \left( \frac{u(t)^2}{\mu^2} - \frac{2\beta^2}{\beta_2\beta_\psi} \right) + \frac{2\beta^2}{\beta_2\beta_\psi} e^{g_\psi[\tau - t]} \right. \\
&\quad \left. \left( e^{-(\beta_\psi - g_\psi)(\tau - t)} - e^{-(\beta_2 - g_\psi)(\tau - t)} \right) \frac{2\beta}{\beta_2 - \beta_\psi} \left( \frac{u(t)}{\mu} - \frac{\beta}{\beta_\psi} \right) \right. \\
&\quad \left. - e^{2g[\tau - t]} \left[ e^{-\beta[\tau - t]} \left( \frac{u(t)}{\mu} - 1 \right) + 1 \right]^2 \right]. \tag{51}
\end{aligned}$$

The structure of the variance is similar to previous structures in e.g. (36) for expected utility. There are growth and convergence rates (50) and there are expected long-run quantities. As a measure of volatility, however, the variance of the growth rate is less suitable for a variety of reasons: First, when we let  $\tau$  become very large, i.e. when we look at the “long run”  $T \gg t$ , we do get a simpler expression as all convergence terms disappear (appendix C.2.3),

$$\frac{var_t(u(T))}{u(t)^2} = \frac{\mu^2}{u(t)^2} \left( \frac{2\beta^2}{\beta_2\beta_\psi} e^{g_\psi[T - t]} - e^{2g[T - t]} \right). \tag{52}$$

This expression, however, represents the variance of the growth rate between  $t$  and  $T$ , i.e. we would not compute the variance of annual growth rates but of  $T - t$ -year growth rates. Clearly, such a variance can never be estimated in reality. Second, the expression for the variance for annual growth rates, i.e. growth rates from  $t$  to  $t+1$ , is the complete expression in (A.50) for  $\tau = t+1$ . Understanding properties of this expression, like derivatives with respect to certain tax rates appears analytically hopeless. Third, as a potential theoretical way out, one could try and deterministically detrend  $u(\tau)$  as discussed on page 16. Computing the variance of the growth rate of deterministically detrended  $u(\tau)$  (and not of  $u(\tau)$  as done here), however, does not yield a finite expression either as the variance grows at  $g_\psi$  while inserting  $e^{-g[\tau - t]}$  in front of  $u(\tau)$  in (49) would not compensate for  $g_\psi$ .

## 7.3 The cyclical component

### 7.3.1 The basic differential equation (41)

As  $\hat{u} = (\hat{C}/L)^{1-\sigma}/(1-\sigma)$  from (39), we have  $d\hat{u} = \frac{(1/L)^{1-\sigma}}{1-\sigma} d\hat{C}^{1-\sigma}$ . With  $\hat{C} = \Psi\hat{K}$  from (19), (25) and the change of variable formula (CVF), we obtain

$$\begin{aligned}
d\hat{C}^{1-\sigma} &= \Psi^{1-\sigma} d\hat{K}^{1-\sigma} \\
&= \left\{ \frac{b_0}{\Psi^{1-\sigma}} \hat{K}^\alpha L^{1-\alpha} - \frac{b_1}{1-\sigma} \hat{K} \right\} (1-\sigma) \Psi^{1-\sigma} \hat{K}^{-\sigma} dt + \left\{ (1 - (1 - A^{-1}\xi))^{1-\sigma} - 1 \right\} \Psi^{1-\sigma} \hat{K}^{1-\sigma} dq \\
&= \left\{ b_0 \hat{K}^{\alpha-\sigma} L^{1-\alpha} (1-\sigma) - b_1 \hat{C}^{1-\sigma} \right\} dt + \left\{ (A^{-1}\xi)^{1-\sigma} - 1 \right\} \hat{C}^{1-\sigma} dq \\
&= \left\{ b_0 \hat{K}^{\alpha-\sigma} L^{1-\alpha} (1-\sigma) - b_1 \hat{C}^{1-\sigma} \right\} dt - \{1 - \Xi/\Theta\} \hat{C}^{1-\sigma} dq.
\end{aligned}$$



Using  $\sigma = \alpha$  from (18), inserting and simplifying yields

$$\begin{aligned} d\hat{u} &= \frac{(1/L)^{1-\sigma}}{1-\sigma} \left[ \left\{ b_0 \hat{K}^{\alpha-\sigma} L^{1-\alpha} (1-\sigma) - b_1 \hat{C}^{1-\sigma} \right\} dt - \{1 - \Xi/\Theta\} \hat{C}^{1-\sigma} dq \right] \\ &= \{b_0 - b_1 \hat{u}\} dt - \{1 - \Xi/\Theta\} \hat{u} dq. \end{aligned}$$

### 7.3.2 Computing moments

The integral version of (41) for  $\tau > t$  is  $\hat{u}(\tau) = \hat{u}(t) + \int_t^\tau (b_0 - b_1 \hat{u}(s)) ds - \int_t^\tau \hat{b}_2 \hat{u}(s) dq(s)$ . Using the martingale result (47), the expected value of  $\hat{u}(\tau)$  is  $E_t \hat{u}(\tau) = \hat{u}(t) + \int_t^\tau (b_0 - b_1 E_t \hat{u}(s)) ds - \lambda \int_t^\tau \hat{b}_2 E_t \hat{u}(s) ds$ . This describes the evolution of the first moment of  $\hat{u}$ . Expressed as a differential equation and using the definition in (42), we obtain  $\dot{\hat{m}}_1(\tau) = b_0 - (b_1 + \lambda \hat{b}_2) \hat{m}_1(\tau)$ . The solution of this deterministic linear differential equation is  $\hat{m}_1(\tau) = e^{-(b_1 + \lambda \hat{b}_2)(\tau-t)} \left( \hat{m}_1(t) + \int_t^\tau e^{(b_1 + \lambda \hat{b}_2)(s-t)} b_0 ds \right) = e^{-(b_1 + \lambda \hat{b}_2)(\tau-t)} \left( \hat{m}_1(t) + b_0 \frac{e^{(b_1 + \lambda \hat{b}_2)(\tau-t)} - 1}{b_1 + \lambda \hat{b}_2} \right)$ , which can be simplified to

$$\hat{m}_1(\tau) = e^{-(b_1 + \lambda \hat{b}_2)(\tau-t)} \left( \hat{m}_1(t) - \frac{b_0}{b_1 + \lambda \hat{b}_2} \right) + \frac{b_0}{b_1 + \lambda \hat{b}_2}. \quad (53)$$

As  $b_1 + \lambda \hat{b}_2 > 0$ , the first moment of  $\hat{u}$  is in the long run given by  $\hat{m}_1(\infty) \equiv \lim_{\tau \rightarrow \infty} \hat{m}_1(\tau) = \frac{b_0}{b_1 + \lambda \hat{b}_2}$ , as presented in (43).

For higher moments, the basic differential equation determining the evolution of  $\hat{u}^n$  is from (41)

$$\begin{aligned} d\hat{u}^n &= n \hat{u}^{n-1} \{b_0 - b_1 \hat{u}\} d\tau - \left\{ 1 - (1 - \hat{b}_2)^n \right\} \hat{u}^n dq \\ &= n \{b_0 \hat{u}^{n-1} - b_1 \hat{u}^n\} d\tau - \left\{ 1 - (1 - \hat{b}_2)^n \right\} \hat{u}^n dq. \end{aligned} \quad (54)$$

Using the integral version, applying expectations and the martingale result (47), we obtain  $dE_t \hat{u}^n = \left\{ nb_0 E_t \hat{u}^{n-1} - \left( nb_1 + \lambda \left[ 1 - (1 - \hat{b}_2)^n \right] \right) E_t \hat{u}^n \right\} dt$ . Using again (42),

$$\dot{\hat{m}}_n = nb_0 \hat{m}_{n-1} - \left( nb_1 + \lambda \left[ 1 - (1 - \hat{b}_2)^n \right] \right) \hat{m}_n. \quad (55)$$

It can now be shown that all moments are constant for  $\tau \rightarrow \infty$ . This follows from (53) for the first moment and from appendix C.3.1 for the 2nd moment. This proof simply inserts (53) into (54) and solves the differential equation. Proofs for higher moments would follow an identical approach. Hence, for the long run where  $\dot{\hat{m}}_n = 0$ , we have from (55)

$$\hat{m}_n(\infty) = \frac{nb_0}{nb_1 + \lambda \left[ 1 - (1 - \hat{b}_2)^n \right]} \hat{m}_{n-1}(\infty). \quad (56)$$

By inserting  $n = 2$ , this directly implies (44), with  $n = 1$ , it becomes (43), remembering that  $\hat{m}_0 = 1$  by definition.

A well-known theorem states that a distribution with limited range is completely characterized by its integer moments (e.g. Casella and Berger, 1990, th. 2.3.3.). As our long-run moments are constant and the range of  $\hat{u}$  is finite, the distribution of  $\hat{u}$  exists, is unique and stationary. Looking at the structure of moments in (56) further shows that the distribution of  $\hat{u}$  is some generalized  $\beta$ -distribution: If  $\hat{b}_2 = 1$ , (56) can be written as  $m_n^c(\infty) = \frac{nb_0}{nb_1 + \lambda} m_{n-1}^c(\infty)$ . Starting from  $\hat{m}_0 = 1$ , repeated inserting yields

$$m_n^c(\infty) = \frac{b_0^n n!}{\prod_{i=1}^n (ib_1 + \lambda)} = \left(\frac{b_0}{b_1}\right)^n \frac{\Gamma(n+1)}{\prod_{i=1}^n (i + \lambda/b_1)} = \left(\frac{b_0}{b_1}\right)^n \frac{\Gamma(n+1)\Gamma(1 + \lambda/b_1)}{\Gamma(n+1 + \lambda/b_1)},$$

where  $\Gamma(\cdot)$  is the gamma-function. The last expression represents, apart from the scaling factor  $(b_0/b_1)^n$ , the  $n$ th moment of a  $\beta$ -distribution with parameters 1 and  $\lambda/b_1$ . Since the  $\beta$ -distribution is determined by its moments, we conclude that, for  $\hat{b}_2 = 1$ ,  $\hat{u}$  has the asymptotic representation  $\hat{u} = \left(\frac{b_0}{b_1}\right)^n X$ , where  $X \sim \text{Beta}(1, \lambda/b_1)$ . With  $\hat{b}_2 \neq 1$ , we obtain a generalized  $\beta$ -distribution which, to the best of our knowledge, has not been encountered before. Analyzing its properties in detail will have to be done in future research.<sup>20</sup>

## References

- Aghion, P., A. Banerjee, and T. Piketty (1999): “Dualism and macroeconomic volatility,” *Quarterly Journal of Economics*, 114, 1359–1397.
- Aghion, P., C. Harris, P. Howitt, and J. Vickers (2001): “Competition, Imitation and Growth with Step-by-Step Innovation,” *Review of Economic Studies*, 68, 467–492.
- Aghion, P., and P. Howitt (1992): “A Model of Growth Through Creative Destruction,” *Econometrica*, 60, 323–351.
- Andolfatto, D., and G. MacDonald (1998): “Technology Diffusion and Aggregate Dynamics,” *Review of Economic Dynamics*, 1, 338–370.
- Arias, A., G. D. Hansen, and L. E. Ohanian (2006): “Why Have Business Cycle Fluctuations Become Less Volatile?,” NBER Working Paper 12079.
- Auerbach, A. J., and J. Slemrod (1997): “The Economic Effects of the Tax Reform Act of 1986,” *Journal of Economic Literature*, 35, 589–632.
- Barlevy, G. (2004): “The Cost of Business Cycles Under Endogenous Growth,” *American Economic Review*, 94, 964 – 990.

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- Benhabib, J., and A. Rustichini (1994): “A note on a new class of solutions to dynamic programming problems arising in economic growth,” *Journal of Economic Dynamics and Control*, 18, 807–813.
- Bental, B., and D. Peled (1996): “The Accumulation of Wealth and the Cyclical Generation of New Technologies: A Search Theoretic Approach,” *International Economic Review*, 37, 687–718.
- Beveridge, S., and C. R. Nelson (1981): “A New Approach to Decomposition of Economic Time Series into Permanent and Transitory Components with Particular Attention to Measurement of the ‘Business Cycle’,” *Journal of Monetary Economics*, 7, 151–174.
- Boldrin, M., and D. Levine (2004): “IER Lawrence Klein lecture: The case against intellectual monopoly,” *International Economic Review*, 45, 327–350.
- Boldrin, M., and D. K. Levine (2001): “Growth Cycles and Market Crashes,” *Journal of Economic Theory*, 96, 13–39.
- Casella, G., and R. L. Berger (1990): *Statistical inference*. Duxbury, Pacific Grove, CA.
- Francois, P., and H. Lloyd-Ellis (2003): “Animal Spirits Trough Creative Destruction,” *American Economic Review*, 93, 530–550.
- Francois, P., and J. Roberts (2003): “Contracting Productivity Growth,” *Review of Economic Studies*, 70, 59–85.
- Funk, P. (1996): “Endogenous Growth, Temporary Equilibrium, and the Direction of Change,” Discussion Paper No A-506, University of Bonn.
- Galí, J., M. Gertler, and J. D. Lopez-Salido (2003): “Markups, Gaps, and the Welfare Costs of Business Fluctuations,” Rev. Version of NBER Working Paper, 8850.
- Gancia, G., and F. Zilibotti (2005): “Horizontal Innovation in the Theory of Growth and Development,” In P. Aghion and S. Durlauf (eds.), *Handbook of Economic Growth* 1, ch. 3. Elsevier.
- García, M. A., and R. J. Griego (1994): “An Elementary Theory of Stochastic Differential Equations Driven by A Poisson Process,” *Commun. Statist. - Stochastic Models*, 10(2), 335–363.

- Grossman, G. M., and E. Helpman (1991): “Quality Ladders in the Theory of Growth,” *Review of Economic Studies*, 58, 43–61.
- Gruber, J. (2006): “A Tax-Based Estimate of the Elasticity of Intertemporal Substitution,” NBER Working Paper 11945.
- Hellwig, M., and A. Irmen (2001): “Endogenous Technical Change in a Competitive Economy,” *Journal of Economic Theory*, 101, 1 – 39.
- Jovanovic, B. (2006): “Asymmetric Cycles,” *Review of Economic Studies*, 73, 145–162.
- Kim, C.-J., and C. R. Nelson (1999): “Has the U.S. Economy Become More Stable? A Bayesian Approach Based on a Markov-Switching Model of the Business Cycle,” *Review of Economics and Statistics*, 81, 608–616.
- Kleiber, C., and S. Kotz (2003): *Statistical Size Distributions in Economics and Actuarial Sciences*. Wiley.
- Long, J. B. Jr., and C. I. Plosser (1983): “Real Business Cycles,” *Journal of Political Economy*, 91, 39–69.
- Lucas, R. E. Jr. (1987): *Models of Business Cycles*. Basil Blackwell Ltd., Oxford.
- Maliar, L., and S. Maliar (2004): “Endogenous Growth and Endogenous Business Cycles,” *Macroeconomic Dynamics*, 8, 559–581.
- Matsuyama, K. (1999): “Growing Through Cycles,” *Econometrica*, 67, 335–347.
- (2001): “Growing through Cycles in an Infinitely Lived Agent Economy,” *Journal of Economic Theory*, 100, 220–234.
- McConnel, M. M., and G. Perez-Quiros (2000): “Output Fluctuations in the United States: What Has Changed Since the Early 1980’s?,” *American Economic Review*, 90, 1464–1476.
- Nelson, C. R., and C. I. Plosser (1982): “Trends and Random Walks in Macroeconomic Time Series: Some Evidence and Implications,” *Journal of Monetary Economics*, 10, 139–162.
- Phillips, K. L., and J. Wrase (2005): “Is Schumpeterian "Creative Destruction" a Plausible Source of Endogenous Real Business Cycles Shocks?,” *Forthcoming Journal of Economic Dynamics and Control*.

- Posch, O. (2006): “Explaining output volatility: The case of taxation,” mimeo University of Hamburg.
- Ramey, G., and V. A. Ramey (1995): “Cross-Country Evidence on the Link Between Volatility and Growth,” *American Economic Review*, 85, 1138–1151.
- Rosenberg, N. (1994): *Exploring the Black Box. Technology, Economics, and History*. Cambridge University Press, Cambridge.
- Segerstrom, P. S. (1998): “Endogenous Growth without Scale Effects,” *American Economic Review*, 88, 1290–1310.
- Sennewald, K. (2006): “Controlled Stochastic Differential Equations under Poisson Uncertainty and with Unbounded Utility,” *Journal of Economic Dynamics and Control*, forthcoming.
- Sennewald, K., and K. Wälde (2005): “Ito’s Lemma” and the Bellman equation for Poisson processes: An applied view,” *Journal of Economics*, forthcoming.
- Stock, J. H., and M. W. Watson (2003): “Has the Business Cycle Changed and Why?,” *NBER Macroeconomics Annual*, 2002.
- Vissing-Jorgensen, A. (2002): “Limited Asset Market Participation and the Elasticity of Intertemporal Substitution,” *Journal of Political Economy*, 110, 825–853.
- Wälde, K. (1999): “Optimal Saving under Poisson Uncertainty,” *Journal of Economic Theory*, 87, 194–217.
- (2005): “Endogenous Growth Cycles,” *International Economic Review*, 46, 867–894.
- Xie, D. (1991): “Increasing Returns and Increasing Rates of Growth,” *Journal of Political Economy*, 99, 429 – 435.
- Zilibotti, F. (1995): “A Rostovian model of endogenous growth and underdevelopment traps,” *European Economic Review*, 39, 1569–1602.

## Appendix

An additional Referees’ appendix is available upon request.