Inequality Constraints in Recursive Economies

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Abstract. Dynamic models with inequality constraints pose a challenging problem for two major reasons: Dynamic Programming techniques often necessitate a non established differentiability of the value function, while Euler equation based techniques have problematic or unknown convergence properties. This paper aims to resolve these two concerns: An envelope theorem is presented that establishes the differentiability of any element in the convergent sequence of approximate value functions when inequality constraints may bind. As a corollary, convergence of an iterative procedure on the Euler equation, usually referred to as time iteration, is ascertained. This procedure turns out to be very convenient from a computational perspective; dynamic economic problems with inequality constraints can be solved reliably and extremely efficiently by exploiting the theoretical insights provided by the paper.

JEL-classification: C61, C63 and C68.

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All errors are my own. Comments and suggestions are welcome.

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1. Introduction

Dynamic models with inequality constraints are of considerable interest to many economists. In microeconomics, and in particular in *consumption theory*, the importance of liquidity constraints is widely recognized (e.g. Deaton 1991). With respect to macroeconomic models of heterogeneous agents, a debt limit is generally a necessary condition for the existence of an ergodic set (see for instance Ljungqvist and Sargent (2004), Aiyagari (1994) and Krusell and Smith (1998)), and models with limited enforcement have recently proven to provide a realistic description of international co-movements (Kehoe and Perri 2002). Additionally, inequality constraints may convey substantial empirical relevance; for instance, employment laws may prohibit firing and lending contracts may prevent bank runs. Foreign direct investments, minimum wages, price regulations, etc. are all examples of potentially binding inequality constraints. Nonetheless, solving dynamic economic models with inequality constraints is generally perceived as challenging: Methods that can handle inequality constraints with ease, generally suffers from the curse of dimensionality, while methods that can moderate this curse have difficulties dealing with such constraints. This paper shows the conditions under which the *n-step value function* for a dynamic problem with inequality constraints is differentiable, and utilizes this result to show how a Euler equation based method can deal with inequality constraints in an easily implementable, efficient and accurate manner.[1]

In the context of discretized Dynamic Programming, dealing with inequality constraints is generally straightforward; the state space is trivially delimited such that any inequality constraint cannot be violated. Nevertheless, discretized Dynamic Programming severely suffers from the curse of dimensionality. To circumvent this difficulty, researchers have on many instances relied upon continuous state approximation methods.[2] These procedures generally work well for interior problems where it is known that the value function is differentiable, which is commonly a necessary condition to recover the equilibrium policy function. However, given that Benveniste and Scheinkman's (1979) envelope theorem assumes interiority, this result does not

[1] The "n-step value function" refers to any element in the sequence $\{v_n\}_{n\in\mathbb{N}}$.

[2] Or, equivalently, "Parameterized Dynamic Programming".

extend to models where inequality constraints may occasionally bind. In the literature, many researchers have chosen to ignore this problem and to proceed as the value function is known to be differentiable even when such constraints are present.

An appealing approach to deal with inequality constraints in dynamic models is to operate on the Euler equation. Christiano and Fisher (2000) show that such constraints can be dealt with in a straightforward way when preferably using the parameterized expectations algorithm developed by den Haan and Marcet (1990), or a version thereof.[3] However, when using such Euler equation based methods, convergence is far from certain and, without an "educated" initial guess for the equilibrium policy function, convergence may indeed often fail.[4]

This paper addresses these concerns. It will be shown that under certain conditions, any element of the sequence of value functions defined by *value function iteration* is differentiable when a general class of inequality constraints are considered. Moreover, analytical expressions of their respective derivatives will be presented.

By exploiting these theoretical insights, an iterative procedure on the Euler equation, commonly known as *time iteration*, is derived. Given that this procedure is equivalent to value function iteration, it is in effect a globally convergent method of finding the equilibrium functions. Due to the concavity of the problem, this turns out to be a very convenient and efficient technique from a computational perspective.

The outline of the paper is the following: Section 2 states and proves the paper's main propositions. Section 3 shows through three examples how the results in section 2 may be implemented in practice. Section 4 concludes.

2. Theory

In this section two central propositions will be presented: Proposition 1 establishes the conditions under which any element of the convergent sequence of approximate value functions, $\{v_n\}_{n\in\mathbb{N}}$, is differentiable. After defining *time iteration* as a particular

[3]See McGrattan (1996) for an alternative Euler equation based technique that utilizes the notion of a "penalty function".

[4]In Christiano and Fisher (2000), a log linearized version of the model is solved and used as an initial guess for the equilibrium functions.

iterative procedure on the Euler equation, Proposition 2 will establish that the sequence of policy functions generated by this method converges to the unique solution.

This paper looks for solutions for problems that may be framed on the basis of the following Bellman equation

$$v(x, z) = \max_{y \in \Gamma(x,z)} \{F(x, y, z) + \beta \int_Z v(y, z')Q(z, dz')\} \quad (1)$$

Where $x \in X$ is the endogenous state, $z \in Z$ is the exogenous state with a law of motion determined by the stationary transition function Q. The following is assumed:

i X is a convex Borel set in $\mathbb{R}^\ell$ with Borel subsets $\mathcal{X}$, and Z is a compact Borel set in $\mathbb{R}^k$ with Borel subsets $\mathcal{Z}$. Denote the (measurable) product space of $(X, \mathcal{X})$ and $(Z, \mathcal{Z})$ as $(S, \mathcal{S})$.

ii The transition function, Q, has the Feller property.[5]

iii The feasibility correspondence $\Gamma(x, z)$ is nonempty, compact-valued, and continuous. Moreover, the set $A = \{(y, x) \in X \times X : y \in \Gamma(x, z)\}$ is convex in x, for all $z \in Z$.

iv The return function $F(\cdot, \cdot, z) : A \to \mathbb{R}$ is, once continuously differentiable, strictly concave and bounded on A for all $z \in Z$.

v The discount factor, β, is in the interval $(0, 1)$.

It is important to note that the above definition of the feasibility correspondence includes the possibility of inequality constraints.

If v_0 is (weakly) concave and the above assumptions hold, the following statements are true for any $n \in \mathbb{N}$ (Section 9.2 in Stokey, Lucas and Prescott 1989):

i The sequences of functions defined by

$$v_{n+1}(x, z) = \max_{y \in \Gamma(x,z)} \{F(x, y, z) + \beta \int_Z v_n(y, z')Q(z, dz')\}$$

$$g_{n+1}(x, z) = \operatorname*{argmax}_{y \in \Gamma(x,z)} \{F(x, y, z) + \beta \int_Z v_n(y, z')Q(z, dz')\}$$

converge pointwise (in the *sup-norm*) to the unique fixed points v and g.[6]

ii v and v_n are strictly concave.

iii g and g_n are continuous functions.

[5] Alternatively one may assume that Z is countable and $\mathcal{Z}$ contains all subsets of Z.

[6] Where g is the argmax of (1).

For subsequent reference, the following additional assumptions will be used

Assumption 1. *The feasibility correspondence can be formulated as*

$$\Gamma(x,z) = \{y \in X : m_j(x,y,z) \leq 0, j = 1,\ldots,r\}$$

and the functions $m_j(x,y,z)$, $j = 1,\ldots,r$, are, once continuously differentiable in x and y, and convex in y.

Assumption 2. *Linear Independence Constraint Qualification (LICQ): The Jacobian of the p binding constraints has full (row) rank; i.e. $rank(\mathbf{J_m}) = p$.*

Assumption 3. *The following hold*

i $\Gamma(x,z) \subset int(X)$ or

ii *X is compact and $g_n(x,z) \in int(X)$.*

Note that Assumption 2 implies that there exists a $\hat{y}$ such that $m_j(x,\hat{y},z) < 0$, for all x, z and j (Slater's Condition). Moreover, part (i) in Assumption 3 implies part (ii), but the converse is not necessarily true.

Define the operator T on $C^1(S)$, the space of bounded, strictly concave once continuously differentiable functions, as

$$(Tf)(x,z) = \max_{y\in\Gamma(x,z)} \{F(x,y,z) + \beta \int_Z f(y,z')Q(z,dz')\} \tag{2}$$

Before moving ahead, it is important to note that under the above additional assumptions it is possible to express the problem in (2) as

$$\begin{aligned} (Tf)(x,z) &= \min_{\mu\geq 0} \max_{y\in X} L(x,y,z,\mu) = \max_{y\in X} \min_{\mu\geq 0} L(x,y,z,\mu) \\ L(x,y,z,\mu) &= F(x,y,z) + \beta \int_Z f(y,z')Q(z,dz') - \sum_{j=1}^{r} \mu_j m_j(x,y,z) \end{aligned} \tag{3}$$

where $L(x,y,z,\mu)$ is a saddle function (see for instance Rockafellar 1970).

The ultimate goal of this section is to show that time iteration yields a convergent sequence of policy functions. The following definition of time iteration will be used.[7]

[7]This definition covers of course the special cases of time iteration discussed in, for instance, Judd (1998), and Coleman (1990). As far as the author is aware, there has been no application of "time iteration" that has not complied with this definition.

Definition 1. *Denote the partial derivatives of F and m with respect to the ith element of y as $F_i(x,y,z)$ and $m_{j,i}(x,y,z)$, respectively. Then,* ***time iteration*** *is the iterative procedure that finds the sequence $\{h_n(x,z)\}_{n=0}^{\infty}$ as $y = h_{n+1}(x,z)$ such that*

$$0 = F_i(x,y,z) + \beta \int_Z [F_i(y, h_n(y,z'), z')$$

$$- \sum_{j=1}^{r} \mu_{j,n}(y,z') m_{j,i}(y, h_n(y,z'), z')] Q(z,dz') - \sum_{j=1}^{r} \mu_{j,n+1}(x,z) m_{j,i}(x,y,z)$$

Notwithstanding the seemingly esoteric notation, time iteration can be thought of as using the Euler equation to find today's optimal policy, h_{n+1}, given the policy of tomorrow, h_n.

In order to verify that this procedure yields a sequence of policy functions converging to g, the following will be shown: Proposition 1 ascertains that the value functions v_n, all $n \in \mathbb{N}$, are differentiable and, by exploiting this finding, Proposition 2 will establish the desired result.

The following lemma is necessary for Proposition 1.

Lemma 1. *The minimizer, $\mu(x,z)$, of* (3) *is a continuous function with respect to x and z.*

Proof. By the definition of a saddle function, the fact that $\mu \geq 0$ and $m_j(x,\hat{y},z) < 0$, for all x, z and j, it follows that

$$(Tf)(x,z) \geq L(x,\hat{y},z,\mu^*) \geq F(x,\hat{y},z) + \beta \int_Z f(\hat{y},z') Q(z,dz') - \mu_j(x,z) m_j(x,\hat{y},z)$$

Which further implies that

$$\mu_j(x,z) \leq \bar{\mu}_j \equiv \max_{x \in X} \frac{(Tf)(x,z) - F(x,\hat{y},z) - \beta \int_Z f(\hat{y},z') Q(z,dz')}{-m_j(x,\hat{y},z)} < +\infty$$

By Berge's Theorem of the Maximum, $L(x, h(x,z,\mu), z, \mu)$ is a continuous function. Hence, the set of minimizers $\mu(x,z)$ that solve the dual problem

$$\min_{0 \leq \mu \leq \bar{\mu}} L(x, \tilde{g}(x,z,\mu), z, \mu)$$

is an upper hemicontinuous correspondence in x and z. By Assumptions 2 and 3, $\mu(x,z)$ is single valued and consequently a continuous function in x and z. □

Proposition 1. *The n-step value function, v_n, is (once) continuously differentiable with respect to $x \in int(X)$ and its partial derivatives are given by*

$$v_{i,n}(x,z) = F_i(x, g_n(x,z), z) - \sum_{j=1}^{r} \mu_{j,n}(x,z) m_{i,j}(x, g_n(x,z), z)$$

for $i = 1, \ldots, \ell$.

Proof. It is sufficient to show that $T : C^1(S) \to C^1(S)$.

Define the saddle function

$$L(x, g(x,z), z, \mu(x,z)) = F(x, g(x,z), z) + \beta \int_Z f(g(x,z), z') Q(z, dz')$$

$$- \sum_{j=1}^{r} \mu_j(x,z) m_j(x, g(x,z), z) = (Tf)(x,z)$$

Pick an $x \in int(X)$ and an x' in a neighborhood, $N_\varepsilon(x)$, of x such that $\| x - x' \| = \| x_i - x_i' \|$ for all $x' \in N_\varepsilon(x)$, where x_i denotes the ith element of the vector x.[8] For notational convenience, denote the policy and multiplier functions from (3) as g, μ and g', μ' for (x,z) and (x',z) respectively.

The definition of a saddle function implies

$$L(x', g, z, \mu') \leq L(x', g', z, \mu') \leq L(x', g', z, \mu)$$

and

$$L(x, g', z, \mu) \leq L(x, g, z, \mu) \leq L(x, g, z, \mu')$$

Combine these two expressions and divide by $x_i' - x_i$

$$\frac{L(x', g, z, \mu') - L(x, g, z, \mu')}{x_i' - x_i} \leq \frac{(Tf)(x', z) - (Tf)(x, z)}{x_i' - x_i}$$

$$\leq \frac{L(x', g', z, \mu) - L(x, g', z, \mu)}{x_i' - x_i}$$

By Lemma 1 and the results on page 4, the functions g and μ are continuous. Consequently the limits of g' and μ' exist and equal $\lim_{x' \to x} g' = g$, $\lim_{x' \to x} \mu' = \mu$. Hence

$$\lim_{x' \to x} \frac{L(x', g, z, \mu') - L(x, g, z, \mu')}{x_i' - x_i} = \lim_{x' \to x} \frac{L(x', g', z, \mu) - L(x, g', z, \mu)}{x_i' - x_i}$$

[8]Where $\| \cdot \|$ denotes the Euclidian norm. This implies that the elements of vectors x and x' are identical except for element i.

By the Pinching (Squeeze) Theorem

$$\lim_{x' \to x} \frac{(Tf)(x', z) - (Tf)(x, z)}{x_i' - x_i} = L_i(x, g, z, \mu)$$

Thus

$$\frac{\partial (Tf)(x, z)}{\partial x_i} = L_i(x, g, z, \mu) = F_i(x, g, z) - \sum_{j=1}^{r} \mu_j m_{j,i}(x, g, z)$$

If v_0 is a weakly concave and differentiable function, the desired result is achieved. □

Note that since the space $C^1(S)$ is not complete in the sup-norm, Proposition 1 does not imply that the limiting value function, v, is differentiable. Moreover, in the proposition above, strict concavity of the problem and full rank of $\mathbf{J_m}$ is assumed. This simplifies the proof given in Corollary 5, p. 597, in Milgrom and Segal (2002), which essentially is equivalent for $x \in [0, 1]$.

The final proposition will show that the sequence of policy functions obtained by time iteration converges to the true policy function.

Proposition 2. *The function $y = h_{n+1}(x, z)$ that solves*

$$0 = F_i(x, y, z) + \beta \int_Z [F_i(y, g_n(y, z'), z')$$

$$- \sum_{j=1}^{r} \mu_{j,n}(y, z') m_{j,i}(y, g_n(y, z'), z')] Q(z, dz') - \sum_{j=1}^{r} \mu_{j,n+1}(x, z) m_{j,i}(x, y, z)$$

for $i = 1, \ldots, \ell$, is equal to

$$g_{n+1}(x, z) = \operatorname*{argmax}_{y \in \Gamma(x,z)} \{F(x, y, z) + \beta \int_Z v_n(y, z') Q(z, dz')\}$$

Proof. Due to the stated assumptions, a sufficient condition for a maximum is a saddle point of the lagrangian

$$L(x, y, z, \mu) = F(x, y, z) + \beta \int_Z v_n(y, z') Q(z, dz') - \sum_{j=1}^{r} \mu_{j,n+1} m_j(x, y, z)$$

By Proposition 1, the value function $v_n(y, z')$ is differentiable and by Assumption 3, given minimizers μ_{n+1}, sufficient conditions for a saddle point are thus[9]

$$0 = F_i(x, y, z) + \beta \int_Z v_{n,i}(y, z') Q(z, dz') - \sum_{j=1}^{r} \mu_{j,n+1}(x, z) m_{j,i}(x, y, z)$$

[9]Assuming that differentiation under the integral is legitimate.

for $i = 1, \ldots, \ell$. By Proposition 1, this can be rewritten as

$$\begin{aligned} 0 =& F_i(x,y,z) + \beta \int_Z [F_i(y, g_n(y,z'), z') \\ &- \sum_{j=1}^{r} \mu_{j,n}(y,z') m_{j,i}(y, g_n(y,z'), z')] Q(z, dz') - \sum_{j=1}^{r} \mu_{j,n+1}(x,z) m_{j,i}(x,y,z) \end{aligned}$$

Due to strict concavity the solution is unique and $h_{n+1}(x,z) = g_{n+1}(x,z)$, which concludes the proof. □

Since it is known that for all $\varepsilon > 0$ there exist an N_s such that $\sup_s |g(s) - g_n(s)| < \varepsilon$ for all $n \geq N_s$, Proposition 2 states that $\sup_s |g(s) - h_n(s)| < \varepsilon$ for all $n \geq N_s$. Hence, the sequence $\{h_n\}_{n \in \mathbb{N}}$ converges to the unique function g.[10]

Lastly, there are two additional remarks to be made: First, $g_n \to g$ implies that $F_i(x, g_n(x,z), z) \to F_i(x, g(x,z), z)$. As a consequence, if $m_j(x,y,z) = m_j(y,z)$, this further implies that $v_{i,n}(x,z) \to F_i(x, g(x,z), z)$.[11] Hence, if convergence of g_n is uniform, then $v(x,z)$ is, under these additional conditions, indeed differentiable and its derivative is given by $F_i(x, g(x,z), z)$. In fact, this result holds under weaker assumptions than previously stated; undeniably, LICQ is dispensable.

Second, a *sufficient* condition for $v(x,z)$ to be differentiable in the more general setting, is that $\mu(s)$ is unique for each $s \in S$.[12]

2.1. **Discussion.** A natural question to ask is how the propositions above are useful in the sense of *finding* the solution to an infinite horizon problem. Indeed, what has been proven is an equivalence between value function and time iteration and, as such, neither method has any advantage over the other. From a strict theoretical viewpoint this is certainly true. However, it should be noted that very few problems actually have an analytical solution, and a numerical approximation to the solution is commonly required. When such procedures are necessary, the propositions above can be used extensively if inequality constraints are present.

[10] If X is compact, N_s is independent of s.

[11] Such constraints, (endogenous) state independent constraints, corresponds, for instance, to debt limits.

[12] If the dual objective function is strictly convex in μ (it is known to be weakly convex), then $\mu(s)$ is unique for each $s \in S$.

To appreciate this line of reasoning, note that in many applications Dynamic Programming relies upon a discretized state space, and such a formulation makes any inequality constraint easy to implement. Nonetheless, to achieve high accuracy the discretization must be made on a very fine grid and this causes the procedure to suffer severely from the curse of dimensionality. To avoid the curse of dimensionality, scholars have relied upon sophisticated approximation methods to enhance accuracy without markedly increasing computer time.[13] Generally, such approximation methods use the derivative of a numerically approximated value function to find the sequence of policy functions. Clearly, Proposition 1 confirms that such continuous state methods will converge to the true solution under a wide set of circumstances.

Moreover, when numerical approximations are used, there may be significant differences between value function- and time iteration, and on some occasions there are reasons to favor the latter: Depending on the character of the problem, the policy function might behave in a less complicated way than the value function, and hence might be more straightforward to approximate. More importantly, given that the derivative of the value function is usually needed to find the policy function, an accurate approximation of its *slope* is as important as its *level*. As a consequence, not only are more data points needed for the approximation, but the choice of approximation method is also restricted. This restriction generally causes Dynamic Programming to suffer more from the curse of dimensionality than time iteration.[14]

As a final remark it ought to be mentioned that time iteration nests "*The method of endogenous gridpoints*" as developed by Carroll (2005). Hence, problems within the preceding framework can thus be solved extremely efficiently with sustained global convergence.

[13]For instance, Judd and Solnick (1994) show, in the case of the standard neoclassical growth model, that using a grid with 12 nodes and applying a shape-preserving spline performs as well as a discretized technique with 1200 nodes.

[14]Approximation methods that are capable of accurately approximating both the level and the slope of a function - certain classes of finite element methods - are not even theoretically developed to deal with high dimensions. Thus, time iteration is the only available technique for reliably solving high-dimensional nonlinear problems.

3. Examples

This section will provide three examples of problems with inequality constraints where time iteration is applicable. The examples are variations of the infinite horizon neoclassical growth model and are chosen on the basis that they represent a large class of models used in the literature. For each respective model, the underlying assumptions required for the results in section 2 will be explicitly verified. In addition, the possible caveats and violations to Assumptions 2 and 3 will be explored.

It is not the purpose of this paper to establish the accuracy or efficiency of various algorithms by solving large scale dynamic programming problems. However, since the first example allows for a closed form solution, an accuracy verification is indeed easily carried out and will thus be presented.

The economies are comprised by an infinite number of ex ante homogenous agents of measure one. The agents maximize their utility by choosing a stochastic consumption process that has to satisfy some feasibility restriction. In general, the problem faced by any agent can be formulated as

$$v(k,z) = \max_{k' \in \Gamma(k,z)} \{u(y(k,z) - k') + \beta \int_Z v(k',z')Q(z,dz')\}$$
$$\Gamma(k,z) = \{k' \in K : m_j(k,k',z) \leq 0, j = 1 \ldots r\}$$

Where $y(k,z) - k'$ denotes consumption, k denotes capital, y is some function determining income and z denotes some stochastic element. Naturally, it is assumed that u, β, K, Z, Q and m fulfill the assumptions stated on page 4. Moreover, it is assumed that $u(c) = \lim_{\gamma \to \sigma} \frac{c^{1-\gamma}}{1-\gamma}$, $\infty > \sigma \geq 1$, and that $y(k,z)$ is concave in k and, unless something else is specifically stated, that y is such that for all $z \in Z$ there exist an $\hat{k} > 0$ such that $k \leq y(k,z) \leq \hat{k}$, all $0 \leq k \leq \hat{k}$, and $y(k,z) < k$, all $k > \hat{k}$. As in most of the neoclassical literature it is assumed that y depends on the function $f(k,h,z) = zk^\alpha h^{1-\alpha}$, for $\alpha \in (0,1)$. Labor, h, is assumed throughout to be supplied inelastically and is normalized to one.

3.1. **An analytical example.** The purpose of this example is to show how the results from Corollary 1 and Propositions 1 and 2 work in a setting with a closed form solution.

It is assumed that $\sigma = 1$, $y(k) = k^\alpha$, $K = [\underline{k}, \overline{k}]$, $m_1(k,k') = b - k'$, $m_2(k,k') = k' - k^\alpha$ and $\alpha \in (0,1)$. The economic model is hence characterized by the Bellman equation

$$v(k) = \max_{k' \in \Gamma(k)} \{\ln(k^\alpha - k') + \beta v(k')\}$$
$$\Gamma(k) = \{k' \in K : b - k' \leq 0, k' - k^\alpha \leq 0\}$$

The model is the deterministic neoclassical growth model with full depreciation and logarithmic utility with an additional constraint on capital holdings. As long as $\underline{k} < b$ and $\overline{k} > 1 = \hat{k}$, Assumption 3 is guaranteed to hold. Note that the specific choice of utility function together with the additional assumption that $0 < b^{1/\alpha} < \underline{k}$ will ensure that $k' - k^\alpha \leq 0$ never is breached. Hence, without violating Assumption 3, it is possible to reduce the correspondence to

$$\Gamma(k) = \{k' \in K : b - k' \leq 0\}$$

By construction Assumption 2 will hold. To eliminate uninteresting cases it is assumed that b is set such that $b < (\frac{1}{\alpha\beta})^{\frac{1}{\alpha-1}}$.

Under the above conditions the results on page 4 hold, and the problem can be solved with value function iteration. Assume for the sake of simplicity that $(b/\beta)^{1/\alpha} < \underline{k} < b$. Then finding

$$v_1(k) = \max_{k' \in \Gamma(k)} \{\ln(k^\alpha - k') + \beta v_0(k')\}$$

for $v_0(k) = \frac{\alpha \ln k + \ln(1-\beta)}{1-\beta}$, corresponds to the time iteration step of finding $k' = g_1(k)$ such that

$$\frac{1}{k^\alpha - k'} + \mu_0(k) = \beta \frac{1}{k'^\alpha - g_0(k')} \alpha k^{\alpha-1}$$

for $g_0(k) = \beta k^\alpha$.[15] Since, the problem itself is strictly concave, it is possible to ignore the multiplier: The policy function from solving this equation is accordingly given by $g_1(k) = \max\{\frac{\alpha\beta}{1-\beta+\alpha\beta} k^\alpha, b\}$. Let $\underline{v}$ and $\overline{v}$ denote the value functions when the agent is and is not constrained respectively. Hence

$$\overline{v}_1(k) = \alpha \frac{1-\beta+\alpha\beta}{1-\beta} \ln k + A_1, \quad \underline{v}_1(k) = \ln(k^\alpha - b) + \beta v_0(b)$$

Where A_1 is some constant. The derivatives of these two functions are given by

$$\overline{v}_1'(k) = \frac{\alpha}{k} \frac{1-\beta+\alpha\beta}{1-\beta}, \quad \underline{v}_1'(k) = \frac{1}{k^\alpha - b} \alpha k^{\alpha-1}$$

[15]Note that $v_0(k) = \frac{\ln(k^\alpha - g_0(k))}{1-\beta}$. Moreover, g_0 is a feasible policy for all $k \in K$. Feasibility of g_0 is not a necessary requirement, but is merely used for the sake of simplicity.

The value function v_1 is consequently differentiable if, and only if, $\overline{v}_1'(k) = \underline{v}_1'(k)$ at k such that $b = \frac{\alpha\beta}{1-\beta+\alpha\beta}k^\alpha$. Inserting this expression for b into $\underline{v}_1'(k)$ yields

$$\underline{v}_1'(k) = \frac{\alpha}{k}\frac{1-\beta+\alpha\beta}{1-\beta} = \overline{v}_1'(k)$$

Hence, v_1 is differentiable and its derivative is given by

$$v_1'(k) = \frac{1}{k^\alpha - g_1(k)}\alpha k^{\alpha-1}$$

Continuing by induction one finds that

$$\begin{aligned}
g_n(k) &= \max\left\{\alpha\beta\frac{(1-\beta)((\alpha\beta)^{n-1}-1)+(\alpha\beta)^{n-1}(\alpha\beta-1)}{(1-\beta)((\alpha\beta)^n-1)+(\alpha\beta)^n(\alpha\beta-1)}k^\alpha, b\right\} \\
\overline{v}_n(k) &= \alpha\ln k\frac{(1-\beta)((\alpha\beta)^n-1)+(\alpha\beta)^n(\alpha\beta-1)}{(1-\beta)(\alpha\beta-1)} + A_n \\
\underline{v}_n(k) &= \ln(k^\alpha - b) + \beta v_{n-1}(b)
\end{aligned}$$

And by the same argument, v_n is differentiable and its derivative is given by

$$v_n'(k) = \frac{1}{k^\alpha - g_n(k)}\alpha k^{\alpha-1}$$

The limiting functions are

$$\begin{aligned}
g(k) &= \max\{\alpha\beta k^\alpha, b\} \\
\overline{v}(k) &= \frac{\alpha}{1-\alpha\beta}\ln k + \frac{\frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta)+\ln(1-\alpha\beta)}{1-\beta} \\
\underline{v}(k) &= \ln(k^\alpha - b) + \beta v(b)
\end{aligned}$$

And the limiting value function is differentiable with derivative

$$v'(k) = \frac{1}{k^\alpha - g(k)}\alpha k^{\alpha-1}$$

Finally, the lagrange multiplier can be recovered as[16]

$$\mu(k) = \frac{1}{k^\alpha - g(k)} - \beta\frac{\alpha g(k)^{\alpha-1}}{g(k)^\alpha - g(g(k))}$$

Since the problem allows for an analytical solution, accuracy of various numerical algorithms can be assessed straightforwardly.

[16]Clearly, the complete sequence of multipliers, $\{\mu_n\}_{n=1}^\infty$, could be recovered in a similar fashion.

TABLE 1. Performance of Algorithms

	Value Iteration		Time Iteration	
Algorithm	#1	#2	#3	#4
N	500	1000	20	20
Accuracy	5.3e-3	3.3e-3	5.8e-4	2.9e-6
REE	4.2e-3	2.1e-3	1.2e-3	3.2e-5
CPU-time	72	295	0.01	0.02
Remark	Discrete grid		Linear	Spline

Table 1 lists the numerical results of applying discretized Value Function Iteration and Time Iteration to the model with $\alpha = 0.3$, $\beta = 1.03^{-1/4}$ $b = 0.15$, $K = [0.7k_{ss}, 1.3k_{ss}]$ and $k_{ss} = (1/\alpha\beta)^{1/(\alpha-1)}$. "*Accuracy*" refers to the maximum absolute value percentage error of the policy function in terms of capital and REE refers to the maximum relative Euler equation errors defined in Judd (1998). Moreover, computer time is

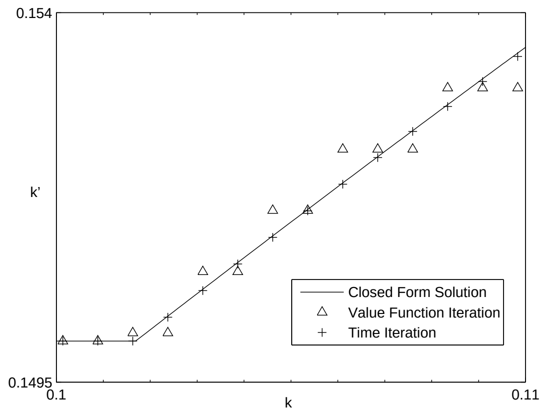

FIGURE 1. Policy functions for Algorithm #1 and #3.

denoted in seconds, *Linear* and *Spline* refer to the interpolation method used for the equilibrium functions and N denotes the number of nodes in the grid. The advantage of time iteration is here quite clear; time iteration outperforms value function iteration in both norms, using a very small grid and in a fraction of the time. The advantage of time iteration is further illuminated by Figure 1 where the policy functions recovered from the procedures are graphed close to the debt limit. Even at the binding point, time iteration performs extremely well.

3.2. **Irreversible investment.** (Christiano and Fisher 2000) Irreversibility of investment in the neoclassical growth model is an important example given that it captures the problem of *state dependent* inequality constraints.

For this economy it is assumed that $y(k,z) = f(k,z) + (1-\delta)k$, $K = [\underline{k}, \overline{k}]$, $m_1(k,k',z) = (1-\delta)k - k'$ and $m_2(k,k',z) = k' - y(k,z)$. Moreover, markets for idiosyncratic risks are complete. The problem is thus characterized by the following Bellman equation

$$v(k,z) = \max_{k' \in \Gamma(k,z)} \{u(y(k,z) - k') + \beta \int_Z v(k',z')Q(z,dz')\}$$
$$\Gamma(x,z) = \{k' \in K : (1-\delta)k - k' \leq 0, k' - y(k,z) \leq 0\}$$

In the previous example, it was possible to use an unbounded return function since the "borrowing constraint" together with restrictions on the income function generated a natural boundedness of the problem. However, in this formulation it is not possible to impose a similar (debt) constraint, since such a restriction would clearly interfere with the irreversibility constraint on investment and hence violate Assumption 2. As an alternative it will be assumed ex ante that there exist an $\varepsilon > 0$ such that for all $z \in Z$, $n \in \mathbb{N}$, $g_n(\underline{k}, z) > \varepsilon$; that is, a lower interiority of $g_n(k,z)$ is ex ante assumed for all k, z and n.[17] By the definition of $\hat{k}$ on page 11, the set of maintainable capital stocks are thus given by $K = [\varepsilon, \hat{k}]$ and, given the specific choice of the utility function, the feasibility correspondence can be reformulated as $\Gamma(x,z) = \{k' \in K : (1-\delta)k - k' \leq 0\}$ without violating Assumption 3.

Under these restrictions it is known that

$$v_{n+1}(k,z) = \max_{k' \geq (1-\delta)k} \{u(y(k,z) - k') + \beta \int_Z v_n(k',z')Q(z,dz')\}$$

converges to v. By Proposition 2 and for a given $\mu_{n+1}(k,z)$, this procedure reduces to finding $k' = g_{n+1}(k,z)$ such that

$$u'(y(k,z) - k') - \mu_{n+1}(k,z) = \beta \int_Z [u'(y(k',z') - g_n(k',z'))y_k(k',z') - \mu_n(k',z')(1-\delta)]Q(z,dz') \tag{4}$$

As can be seen from (4), the multiplier from the previous iteration is in the expectation term. This indicates the presence of a state dependent constraint.

[17] Naturally, such a conjecture needs to be verified when solving the model.

Although it is necessary to find both a policy function and a multiplier at each iteration, this is a trivial task. Since the problem itself is strictly concave, it is possible to ignore μ_{n+1} in (4) and find the function $\hat{g}_{n+1}$ that solves the (reduced) equation. The true policy function g_{n+1} can then be recovered as $g_{n+1} = \max\{\hat{g}_{n+1}, (1-\delta)k\}$ and μ_{n+1} is merely the residual in (4) when g_{n+1} is inserted into the equation.

For a parameterization given by $\alpha = 0.3$, $\beta = 1.03^{-1/4}$, $\delta = 0.02$, $\sigma = 1$, $Z = \exp(\{0.23, -0.23\})$ and $Q(z, z') = 1/2$ for all (z, z') pairs, the solution is depicted in Figure 2. Figure 2 illustrates how distinctly the procedure captures the Kuhn-Tucker condition of $\mu(k, z)m_1(k, k', z) = 0$. The MATLAB program for this model, presented

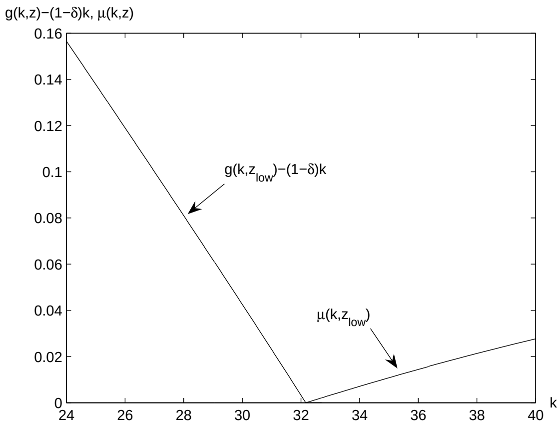

FIGURE 2. Investment function and multiplier for the model with irreversible investment.

in Appendix A, clearly illustrates the simplicity of the procedure.

3.3. **Incomplete markets.** (Aiyagari 1994) Standard models with incomplete market are relevant for the procedure proposed in this paper since the assumption of risk-free borrowing induces a debt limit as a necessary condition for the characterization of the economy to be valid.

It is assumed that $y(k, z) = wz + (1+r)k$, $K = [\underline{k}, \overline{k}]$, Z is countable, $m_1(k, k', z) = -\phi - k'$ and, as before, $m_2(k, k', z) = k' - y(k, z)$. Here z denotes an uninsurable idiosyncratic component; markets are incomplete. However, there is no aggregate risk in the economy.[18] Moreover, w and r are given by $f_h(\tilde{k}, h)$ and $1 + f_k(\tilde{k}, h) - \delta$

[18]See Krusell and Smith (1998) for an economy where this assumption is relaxed.

respectively. $\tilde{k}$ represents the aggregate capital stock in the economy and, as before, h represent the employment rate, normalized to unity. The problem is thus characterized by the following equations

$$v(k,z) = \max_{k' \in \Gamma(k,z)} \{u(y(k,z) - k') + \beta \int_Z v(k',z')Q(z,z')\}$$
$$\Gamma(k,z) = \{k' \in K : -\phi - k' \leq 0, k' - y(k,z) \leq 0\}$$
$$\tilde{k} = \sum_z \int k\lambda(k,z)\, dk$$
$$\lambda(k',z') = \sum_z \int_{\{k:k'=g(k,z)\}} \lambda(k,z)Q(z,z')\, dk$$

Where that $\lambda(k,\varepsilon)$ denotes the (stationary) distribution of asset holdings and employment status.

Note that $y(k,z)$ does not fulfill the desired properties to ensure an upper bound on the endogenous state space (as stated on page 11). However, as noted in Aiyagari (1994), for all $z \in Z$ there exist a k^* such that for all $k \geq k^*$, $k' \leq k$. In order to ensure that Assumption 3 holds, set $\overline{k} > k^*$ and $\underline{k} < -\phi < w\underline{z} + \underline{k}(1+r)$, where $\underline{z} = \inf Z$. By again exploiting the properties of the functional form of the return function, the feasibility correspondence can be reformulated as $\Gamma(k,z) = \{k' \in K : -\phi - k' \leq 0\}$ and Assumption 2 will, by construction, hold.[19]

Under the above stated conditions, it is known that the procedure

$$v_{n+1}(k,z) = \max_{-\phi \leq k'} \{u(y(k,z) - k') + \beta \int_Z v_n(k',z')Q(z,z')\}$$

converges to v. Given $\mu_{n+1}(k,z)$, Proposition 2 asserts that this procedure reduces to finding $k' = g_{n+1}(k,z)$ such that

$$u'(y(k,z) - k') - \mu_{n+1}(k,z) = \beta \int_Z u'(y(k',z') - g_n(k',z'))(1+r)Q(z,z')$$

As in the previous example, it is possible due to the concavity of the problem, to ignore the multiplier μ_{n+1} and solve the problem to find $\hat{g}_{n+1}$. Again, the true policy function g_{n+1} is recovered as $g_{n+1} = \max\{-\phi, \hat{g}_{n+1}\}$. The multiplier can then be obtained as a residual. Thus, except for a applying a "max" operator at each iteration,

[19]Note that $-\phi$ in the above analysis is set strictly *higher* than what Aiyagari (1994) refers to as "the natural debt limit". Here, $-\phi$ is what is usually referred to as an "ad-hoc constraint"; an important feature in the current setting to ensure the boundedness of the problem. See for instance Krusell and Smith (1997) for the empirical relevance of ad-hoc constraints.

such a procedure is no more difficult to solve than a model with no constraints at all.

For a parameterization given by $\alpha = 0.3$, $\beta = 0.95$, $\delta = 0.1$, $\sigma = 1$, $\phi = -2$, $Z = \{1, 0.5\}$ and $Q(z, z') = 1/2$ for all (z, z') pairs, the solution is depicted in Figure 3. Again, Figure 3 illustrates how ably the procedure captures the Kuhn-Tucker condition of $\mu(k, z)m_1(k, k', z) = 0$.

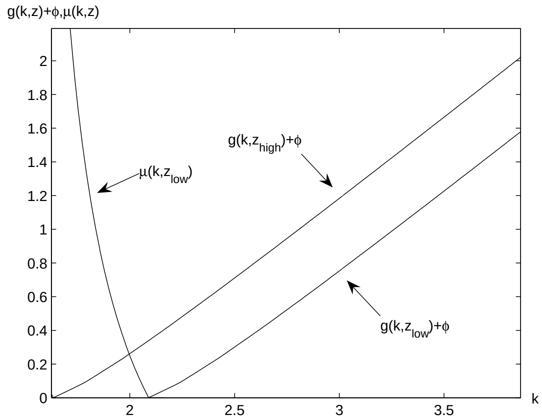

FIGURE 3. Policy and multiplier for an Aiygari economy with an ad hoc constraint ($\phi = -2$).

4. Conclusion

Recursive models with inequality constraints are generally problematic to solve: Discretized Dynamic Programming suffers severely from the curse of dimensionality and Parameterized Dynamic Programming imposes a differentiability property of the value function that might be false. Furthermore, Euler equation techniques have unknown or very poor convergence properties, and are thus difficult to solve without making initial educated guesses for the equilibrium functions.

This paper has resolved parts of these problems: It has been established that under weak conditions, the n-step value function is differentiable for problems with inequality constraints. Thus, solution techniques that impose a differentiability of the value function will, at least theoretically, converge to the true solution. Moreover, through a derived analytical expression of the derivative of the value function, an iterative Euler equation based method has been shown to be convergent when inequality constraints might be present.

Moreover, as shown in section 3, time iteration proposes an iterative procedure that is appealing from a computational perspective. Firstly, high-dimensional approximation methods are applicable given that there is no need to approximate the *slope* of any equilibrium function. Secondly, policy functions possibly have a relatively uncomplicated behavior relatively to the value function and are hence more accurately approximated. Thirdly, in the iterative procedure, lagrange multipliers come out as residuals from the Euler equation and these are, in the case of state dependent constraints, merely needed to be interpolated at each iteration.

As a direction for future research, it would be desirable to establish under which additional conditions the limiting value function is differentiable when inequality constraints potentially bind. Moreover, methods for evaluating the accuracy of numerical solutions using the Euler equation residuals, are well developed for *interior* problems (Santos 2000). However, they are not extended to deal with problems formulated in the context of this paper.

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Appendix A. Matlab code

```
% The neoclassical growth model with irreversible investment
% in the setting of Christiano and Fischer (2000), model (1),
% solved by the method of endogenous gridpoints using a finite
% element method (linear interpolation is default).

% Parameters: exp(z) is the solow residual, a is the capital share
% of output, b is the discount factor, d is the depreciation
% rate and g is the coefficient of relative riskaversion.
% Z is the exogenous state space with associated transition
% matrix, Q.

%N defines the number of nodes in the endogenous state space.

N=200; p=0; z=0.23; a=0.3; b=1.03^(-1/4); d=0.02; g=1;
Q=[(1+p)/2,(1-p)/2;(1-p)/2,(1+p)/2]; Z=exp([z;-z]);

n=ones(size(Z')); nn=ones(N,1); d1=0.5;
khat=((1-b*(1-d))/(a*b))^(1/(a-1)); kmax=khat*1.9;
kmin=khat*0.3;
kp=(linspace(kmin,kmax,N))'; kpp=(1-d)*kp*n; mp=0; mup=0*nn*n;
m0=(kp./(1-d)).^a*Z';

while d1>1e-8
    up=(kp.^(a)*Z'+(1-d)*kp*n-max(kpp,(1-d)*kp*n)).^(-g);
    r=a*kp.^(a-1)*Z'-d;
    m=(b*(up.*(1+r)-max(mup,0))*Q').^(-1/g)+kp*n;
    mu=(m0).^(-g)-b*(up.*(1+r)-max(mup,0))*Q';
    d1=max(max(abs(mp-m)./(1+abs(m))));
    mp=m;
    for i=1:length(Z)
        kpp(:,i)=interp1(m(:,i),kp,Z(i)*kp.^a+(1-d)*kp);
        mup(:,i)=interp1(m(:,i),mu(:,i),Z(i)*kp.^a+(1-d)*kp);
    end
end
```