Being serious about non-commitment: subgame perfect equilibrium in continuous time

Ivar Ekeland*and Ali Lazrak†

Preliminary version: May 2006

Abstract

This paper characterizes differentiable subgame perfect equilibria in a continuous time intertemporal decision optimization problem with non-constant discounting. The equilibrium equation takes two different forms, one of which is reminescent of the classical Hamilton-Jacobi-Bellman equation of optimal control, but with a non-local term. We give a local existence result, and several examples in the consumption saving problem. The analysis is then applied to suggest that non constant discount rates generate an indeterminacy of the steady state in the Ramsey growth model. Despite its indeterminacy, the steady state level is robust to small deviations from constant discount rates.

*University of British Columbia, mathematics and economic departments (ekeland@math.ubc.ca)

†Sauder School of Business at the University of British Columbia (ali.lazrak@sauder.ubc.ca)

1 Introduction

This paper adresses the problem of time inconsistency under non-constant discounting. Whereas our method and result are quite general in character, we have chosen to illustrate them in the framework of the Ramsey model of economic growth, (1928; see for instance [5]), which has also been used as a test case by Barro [4] and Karp [12] in their investigations of the subject.

In its typical formulation, the Ramsey model represents the decision maker as maximizing:

$$\max \int_0^T h(t)u\left(c\left(t\right)\right)dt + h\left(T\right)g\left(k\left(T\right)\right) \tag{1}$$

$$\frac{dk}{dt} = f\left(t, k\left(t\right)\right) - c\left(t\right), \quad k\left(0\right) = k_0 \tag{2}$$

where $[0, T]$ is the life span of the decision maker and where the function f maps $[0, T] \times R^d$ onto R^d. Here, the decision maker can be interpreted either as an individual or as a governement. In either case, c denotes the consumption of the representative individual, $u\left(c\left(t\right)\right)$ is the utility of current consumption, $k\left(t\right)$ is current capital and $g(k(T))$ is the utility of terminal capital. If the decision maker is an individual, $f(t, k(t))$ represents capital rental interest and wages, and if it is a government, it represents production and capital depreciation.

$h : [0, \infty] \to R$ is the discount function. Here and in the sequel, it will be assumed that it is continuously differentiable, with $h\left(0\right) = 1$, $h\left(t\right) \geq 0$, $h'\left(t\right) \leq 0$ and $h\left(t\right) \to 0$ when $t \to \infty$. The classical case, the one considered by Ramsey and the subsequent litterature until the pionneering work of Strotz [25], is the one when the discount rate is constant: $h\left(t\right) = \exp\left(-\rho t\right)$.

The decision maker, be it an individual or a government, faces this maximization problem at time 0, and decides on an overall solution, $t \to \left(\bar{c}\left(t\right), \bar{x}\left(t\right)\right)$, valid for $0 \leq t \leq T$. At any intermediate time t, the decision-maker, either herself at a later time if she is an individual, or whoever is in office if it is a governement, will face a similar problem, namely:

$$\max \int_t^T h\left(t - s\right)u\left(c\left(s\right)\right)ds + h\left(T - t\right)g\left(k\left(T\right)\right)$$

$$\frac{dk}{ds} = f\left(s, k\left(s\right)\right) - c\left(s\right), \quad s \geq t, \quad k\left(t\right) = k_t$$

where k_t is the existing capital at time t. The solution to this problem will be some $s \to \left(\tilde{c}\left(s\right), \tilde{x}\left(s\right)\right)$, valid for $t \leq s \leq T$. If this is different from $s \to \left(\bar{c}\left(s\right), \bar{x}\left(s\right)\right)$, then the decision-maker at time t is being asked to implement a policy which, from her point of view is suboptimal. This she will not do, unless the decision-maker at time 0 has found a way to commit her. If this is not the case, then the optimal policy $t \to \left(\bar{c}\left(t\right), \bar{x}\left(t\right)\right)$ for problem (1) (2) cannot be implemented. This is the problem of *time-inconsistency*, which has been studied by many authors: see [9] for a survey.

It has been known for a long time that in the case where the discount rate is constant, so that $h(t) = \exp(-\rho t)$, time consistency obtains: $(\tilde{c}, \tilde{x}) = (\bar{c}, \bar{x})$, so that the decision-maker at time 0 can count on the decision-makers at all intermediate times to implement the decisions she has planned. The fundamental reason for which time consistency obtains is that preference reversals due to the mere passage of time are precluded: with a constant discount rate, relative preference between two prospective consumption plans is unaffected by their distance into the future[1]. In other words, when the discount function is exponential, relative preferences induced from the discounted utility model do not change with time.

But why should the discount function be precisely exponential ? Experimental evidence from psychology challenges the main consequences to be derived from constant discount rates: see Ainslie [2] and Frederick *et al* [7] for an overview. Relative preferences do seem to change with time. In particular, there is robust evidence of an inclination for imminent gratification even if accompanied by harmful delayed consequences. This suggest a discount rate which is declining over time (see Ainslie [1] and Lowenstein and Prelec [16]). In other words, the discount function h should be *hyperbolic*, that is, h'/h should be decreasing.

In this paper, we will deal with general discount functions: they need not be hyperbolic, but they are certainly not exponential. Then time-inconsistency obtains. We shall also assume that the decision-maker at time 0 cannot commit the decision-makers at later times $t > 0$. This means that the solution of problem (1), (2) cannot be implemented. In other words, there is no way for the decision-maker at time 0 to achieve what is, from her point of view, the optimal solution of the problem, and she must turn to a second-best policy. Defining and studying such a policy is the first aim of this paper. The path to follow is clear. The best the decision-maker at time t can do is to guess what her successors are planning to do, and to plan her own consumption $c(t)$ accordingly. In other words, we will be looking for a subgame-perfect equilibrium of a certain game.

A second idea now comes into play: we will assume that none of the decision-makers is sufficiently powerful to influence the global outcome. This is very similar to perfect competition, where no agent is sufficiently important to affect prices, and it will be formalized in the same way. In his seminal paper [3], Aumann captures that idea by considering an exchange economy where the set of traders is the interval $[0,\ 1]$. An allocation then is a map $x : [0,\ 1] \to R_+^n$

[1] To see this, assume $0 < r < s < t < T$ and consider two consumption plans $c(v)$ and $\bar{c}(v)$ valid for $v \geq t$. The incremental utilities (IU)for self "r" and self "s" are related by

$$\begin{aligned} IU_r : &= \int_t^T e^{-\rho(v-r)} \left[u\left(c(v)\right) - u\left(\bar{c}(v)\right) \right] dv \\ &= e^{-\rho(s-r)} \int_t^T e^{-\rho(v-s)} \left[u\left(c(v)\right) - u\left(\bar{c}(v)\right) \right] dv =: e^{-\rho(s-r)} IU_s \end{aligned}$$

and therefore the ordinal ranking of c and $\bar{c}$ does not change with the mere passage of time.

and the total consumption of a coalition $A \subset [0,\ 1]$ is the integral:

$$\int_A x(t)\,dt$$

so that individuals, and more generally coalitions with vanishing Lebesgue measure, have zero consumption, and therefore cannot influence prices. However, a small coalition $[t,\ t+\varepsilon]$ will be able to do so, and its weight will be roughly proportional to ε.

Similarly, we will consider that the set of decision-makers is the interval $[0,\ T]$. At time t, there is a decision-maker who decides what current consumption $c(t)$ shall be. As is readily seen from the equation (2), changing the value of c at just one point in time will not affect the trajectory. However, the decision-maker at time t is allowed to form a coalition with her immediate successors, that is with all $s \in [t,\ t+\varepsilon]$, and we will derive the definition of an equilibrium strategy by letting $\varepsilon \to 0$. In fact, we are assuming that the decision-maker t can commit her immediate successors (but not, as we said before, her more distant ones), but that the commitment span is vanishingly small.

In section 2, we use that idea to derive a suitable concept of equilibrium strategy. Given a strategy $c = \sigma(t,k)$, a coalition $[t,\ t+\varepsilon]$ will be able to perturb the discounted utility at time t by deviating unilaterally, that is, by choosing some $c(t)$ different from $\sigma(t,k(t))$; the perturbation will of course be of the first order in ε. If there is no incentive for this coalition to deviate, in the sense that this perturbation is always non-positive, and zero if and only if $c(t) = \sigma(t,k(t))$, then σ is an ε-equilibrium, in fact a subgame perfect equilibrium. Letting $\varepsilon \to 0$, we derive an appropriate notion of equilibrium strategy in the case when individual decision makers do not have market power.

In section 3, we characterize the newly defined equilibrium strategies in terms of a value function $V(t,k)$. This function is seen to satisfy two equivalent equations, (IE) and (DE), the latter being very similar to the usual Hamilton-Jacobi-Bellman (HJB) equation of optimal control, and reducing to (HJB) in the case when $h(t)$ is an exponential. However, (DE) is not a partial differential equation: it contains a non-local term, which makes it much more difficult to study than a straightforward partial differential equation. We have only a local existence result, which is stated without proof. However, in section 4, we provide explicit examples in the case when the horizon is infinite, $T=\infty$, and $f(t,k)$ takes the special form $r(t)\,k + w(t)$ (capital revenue plus wage). We also investigate the naive strategy, where each decision-maker simply forgets that he cannot commit his successors, and plays as if she could; we show that it is not an equilibrium strategy, unless $u(c) = \ln c$.

In section 5, we focus on the infinite-horizon problem, with $n=1$, and we investigate whether there is some $\bar{k}$ such that all paths $k(t)$ converge to $\bar{k}$ in equilibrium. This is the question of balanced growth, which has been much studied in the case when $h(t) = \exp(-\rho t)$, and optimal control theory applies; it is well known that in that case, we must have $f'\left(\bar{k}\right) = r$, which effectively pins down the value of $\bar{k}$. In the case of general discount function, we find that $f'\left(\bar{k}\right)$

must belong to some interval, and that, *ceteris paribus*, this interval converges to the point r if $h(t)$ converges to $\exp(-\rho t)$. We conclude in section 6.

The results obtained in sections 5 and 6 are very similar to those obtained earlier by Barro [4] and by Karp [12]. The main contribution of the present paper lies elsewhere, in the precise definition of equilibrium strategies, and in their characterization through a value function $V(t,k)$ which has to satisfy certain equations, reminescent of the (HJB) equation. This allows us to carry the calculations somewhat further than Barro or Karp, and it also opens the door to a systematic study of the problem. The local existence result which we give is an example of what can be obtained through our approach, and not otherwise.

The case when time is discrete, $t_1 = 0, t_2, ..., t_{n-1}, t_n = T$, has been investigated by many authors, for instance Strotz [25] , Pollak [22], Peleg and Yaari [20], Phelps and Pollak [21], and more recently, Laibson [14]. The last decision-maker operates at time t_{n-1}; after he has acted, the party is over. He is facing a plain vanilla optimization problem, and solves it. His predecessor operates at time t_{n-2}. She is faced with a leader-follower game, which she solves by integrating the strategy of her successor into her own decision. In principle, by proceeding recursively in this way, one can go all the way back to $t_1 = 0$, the very first decision to be made (which, again, would not be the optimal one from the time 0 perspective, if this particular decision-maker could commit all her successors). If this method is successful, it yields a subgame perfect equilibrium, and the corresponding policy will follow through despite the lack of commitment devices. It is also important to observe that the equilibrium policy, as in the prisoner's dilemma game, is suboptimal relative to the outcome that can occur with a pre-commitment technology. Using this approach (and extending it), a recent literature has flourished showing that apparent irrationality of individuals, even in financial markets, can be ascribed to the fact that the psychological discount factor is not exponential; see Laibson [15], O'Donoghue and Rabin [19], Harris and Laibson [8], Krusell and Smith [13], Diamond and Koszegi [6], Luttmer and Mariotti [17] and others.

Unfortunately, such games typically fail to have a subgame-perfect equilibrium. The reason is that, even if u is concave with respect to c, the payoff to the decision-maker at time t_i is not concave with respect to his own consumption c_i, because c_i determines the capital k_{i+1} at time t_{i+1}, and constrains the choice of the next decision-maker in a complicated, and certainly non-linear, way. Proceeding recursively from t_{n-1}, the strategy $c_i = \sigma_i(k_i)$ at time t_i will end up being discontinuous with respect to k_i, which effectively kills the hope of finding a subgame-perfect equilibrium. It is a fundamental difficulty of the discrete time model, and various ways have been devised to get around this problem, such as adding a public correlation device, as in Harris, Reny and Robson [23] (see also [24]). With this in mind, it comes as no surprise that existence results for subgame-perfect equilibria in continuous time are so hard to prove.

2 Equilibrium strategies: definition

We consider an intertemporal decision problem where the decision-maker at time t is striving to maximise:

$$\int_t^T h(s-t)\,u(c(s))\,ds + h(T-t)\,g(k(T)) \tag{3}$$

subject to:

$$\frac{dk}{ds} = f(s,k(s)) - c(s)\,, \quad k(t) = k_t. \tag{4}$$

Notations are as stated in the introduction. Recall that h is continuously differentiable, with $h(0)=1$, $h(t)\geq 0$, $h'(t)\leq 0$, and $h(t)\to 0$ when $t\to\infty$. It will also be assumed that u, f, g are twice continuously differentiable, that u is strictly concave, and that f is strictly concave with respect to k.

We shall denote by $i : R^d \to R^d$ the inverse of the derivative $u' : R^d \to R^d$:

$$u'(c) = x \Longleftrightarrow c = i(x)$$

and it will be assumed that it is continuously differentiable. We shall also consider the Legendre-Fenchel transform $\tilde{u}$ of the concave function u, defined by:

$$\tilde{u}(x) = \max_{c\in R^d}\left(u(c) - xc\right) = u(i(x)) - xi(x)$$

Note that it is a convex function. By the envelope theorem, we have:

$$\tilde{u}'(x) = -i(x) = \left[-u'\right]^{-1}(x)$$

We now proceed to define subgame-perfect equilibrium strategies, using the approach outlined in the introduction. A strategy $c = \sigma(t,k)$ has been announced an is public knowledge. All decision-makers up to time t have applied this strategy, that is, the dynamics of capital between times 0 and t are given by:

$$\frac{dk}{ds} = f(s,k) - \sigma(s,k)\,, \quad k(0) = k_0 \tag{5}$$

The decision-maker at time t inherits a capital k_t, which is the value at $s=t$ of the solution to the Cauchy problem (5). She can commit all the decision-makers in $[t,\ t+\varepsilon]$,where $\varepsilon > 0$ is vanishingly small. She expects all later ones to apply the strategy $\sigma : [0,T]\times R^d \to R^d$, and she asks herself if it is in her own interest to apply the same strategy, that is, to consume $\sigma(t,k)$. If she consumes another bundle, c say, the immediate utility flow during $[t,\ t+\varepsilon]$ is $u(c)\,\varepsilon$. At time $t+\varepsilon$, the resulting capital will be $k + (f(t,k) - c)\,\varepsilon$, and from then on, the strategy σ will be applied. The consumption at time $s \geq t+r$ is $c(s) = \sigma(s,k(s))$, where

$$\frac{dk}{ds} = f(s,k(s)) - \sigma(s,k(s))\,, \quad s\geq t+\varepsilon \tag{6}$$

$$k(t+\varepsilon) = k_t + (f(t,k_t) - c)\,\varepsilon \tag{7}$$

Denote by $k_0(s)$ the future path of capital if the decision-maker at time t applies the strategy σ, that is, if $c = \sigma(t, k_t)$. The dynamic of k_0 is given by:

$$\frac{dk_0}{ds} = f(s, k_0(s)) - \sigma(s, k_0(s)), \quad s \geq t \tag{8}$$

$$k_0(t) = k_t \tag{9}$$

Write $k(s) = k_0(s) + k_1(s)\varepsilon$, plug that into (6),(7), keeping only terms of first order in ε. We get:

$$\begin{aligned}
\frac{dk_0}{ds} + \varepsilon\frac{dk_1}{ds} &= f(s, k_0(s)) + \varepsilon\frac{\partial f}{\partial k}(s, k_0(s))\, k_1(s) - \sigma(s, k_0(s)) \\
&\quad - \varepsilon\frac{\partial \sigma}{\partial k}(s, k_0(s))\, k_1(s), \quad s \geq t + \varepsilon, \\
k(t+\varepsilon) &= k_0(t+\varepsilon) + \varepsilon k_1(t+\varepsilon) \\
&= k_0(t) + \varepsilon\frac{dk_0}{ds}(t) + \varepsilon k_1(t+\varepsilon) \\
&= k_t + \varepsilon\left(f(s, k_t) - \sigma(t, k_t)\right) + \varepsilon k_1(t+\varepsilon)
\end{aligned}$$

where $\frac{\partial f}{\partial k}$ and $\frac{\partial \sigma}{\partial k}$ stand for the matrix of partial derivatives of f and σ with respect to $k \in R^d$. Comparing with (8),(9) and (7), we get the linear differential system:

$$\begin{aligned}
\frac{dk_1}{ds} &= \left(\frac{\partial f}{\partial k}(s, k_0(s)) - \frac{\partial \sigma}{\partial k}(s, k_0(s))\right) k_1(s), \ s \geq t + \varepsilon \\
k_1(t+\varepsilon) &= \sigma(t, k_t) - c
\end{aligned}$$

Summing up, we find that the total gain for the decision-maker at time t from consuming bundle c during the interval of length ε when she can commit, is:

$$\begin{aligned}
&u(c)\varepsilon + \int_{t+\varepsilon}^{T} h(s-t)\, u(\sigma(s, k_0(s) + \varepsilon k_1(s)))\, ds + h(T-t)\left(g(k_0(T) + \varepsilon k_1(T))\right) \\
&= \int_t^T h(s-t)\, u(\sigma(s, k_0(s)))\, ds + h(T-t)\, g(k_0(T)) \\
&+ \varepsilon\left[u(c) - u(\sigma(t,k)) + \int_t^T h(s-t)\frac{\partial u}{\partial c}(\sigma(s, k_0(s)))\frac{\partial \sigma}{\partial k}(s, k_0(s))\, k_1(s)\, ds\right. \\
&\left. + \, h(T-t)\frac{\partial g}{\partial k}(k_0(T))\, k_1(T)\right] + \text{ h.o.t}
\end{aligned}$$

where $\frac{\partial g}{\partial k}$ (resp. $\frac{\partial u}{\partial c}$) is the vector of partial derivatives of g (resp. u) with respect to $k \in R^d$ (resp. $c \in R^d$) and h.o.t denotes higher-order terms in ε.

In the limit, when $\varepsilon \to 0$, and the commitment span of the decision-maker vanishes, we are left with two terms only. Note that the first term does not depend on the decision taken at time t, but the second one does. This is the one that the decision-maker at time t will try to maximize. In other words, given that a strategy σ has been announced and that the current state is $k_t = k$, the decision-maker at time t faces the optimisation problem:

$$\max_c P_1 (t, k, \sigma, c) \tag{10}$$

where:

$$\begin{aligned} P_1 (t, k, \sigma, c) &= u (c) - u (\sigma(t, k)) \\ &+ \int_t^T h (s - t) \frac{\partial u}{\partial c} (\sigma (s, k_0 (s))) \frac{\partial \sigma}{\partial k} (s, k_0 (s)) k_1 (s) ds \\ &+ h (T - t) \frac{\partial g}{\partial k} (k_0 (T)) k_1 (T) , \end{aligned}$$

In the above expression, $k_0 (s)$ and $k_1 (s)$ are given by:

$$\begin{aligned} \frac{dk_0}{ds} &= f (s, k_0 (s)) - \sigma (s, k_0 (s)) \\ k_0 (t) &= k \end{aligned}$$

$$\frac{dk_1}{ds} = \left(\frac{\partial f}{\partial k} (s, k_0 (s)) - \frac{\partial \sigma}{\partial k} (s, k_0 (s)) \right) k_1 (s) \tag{11}$$

$$k_1 (t) = \sigma (t, k) - c \tag{12}$$

Definition 1 *We shall say that $\sigma : [0, T] \times R^d \to R^d$ is an* ***equilibrium strategy for the intertemporal decision model*** *(3),(4) if, for every $t \in [0, T]$ and $k \in R^d$, the maximum in problem (10) is attained for $c = \sigma (t, k)$:*

$$\sigma (t, k) = \arg \max_c P_1 (t, k, \sigma, c)$$

The intuition behind this definition is quite simple. Each decision-maker can commit only for a small time ε, so he can only hope to exert a very small influence on the final result. In fact, if the decision-maker at time t plays c when he/she is called to bat, while all the others are applying the strategy σ, the end payoff for him/her will be of the form

$$P_0 (t, k, \sigma) + \varepsilon P_1 (t, k, \sigma, c)$$

where the first term of the right hand side does not depend on c. In the absence of commitment, the decision-maker at time t will choose whichever c maximizes the second term $\varepsilon P_1 (t, k, \sigma, c)$. Saying that σ is an equilibrium strategy means that the decision maker at time t will choose $c = \sigma (t, k)$, that is, that the strategy σ can be implemented even in the absence of commitment.

Conversely, is a strategy σ for the intertemporal decision model (3),(4) is not an equilibrium strategy, then it cannot be implemented unless the decision-maker at time 0 has some way to commit his successors. Typically, an optimal strategy will not be an equilibrium strategy. More precisely, a strategy which appears to be optimal at time 0 no longer appears to be optimal at times $t > 0$, which means that the decision-maker at time t feels he can do better than whatever was planned for him to do at time 0. In the case of macroeconomic policy, for instance, successive governments will disagree on what is an optimal strategy, even if they agree on the collective utility $u(c)$, so that the concept of equilibrium strategy seems far more reasonable - at least it stands a chance of being implemented.

What happens then if successive decision-makers take the myopic view, and each of them acts as if he could commit his successors ? At time t, then, the decision-maker would maximise the integral (3) with the usual tools of control theory, thereby deriving a consumption $c = \sigma_n(t, k)$. This is the *naive strategy*; in general it will not be an equilibrium strategy, so that every decison-maker has an incentive to deviate. It will be studied in more detail in section 4.

3 Characterization and existence of equilibrium strategies

In this section, we characterize equilibrium strategies of problem (3), (4), by an equation, which we call the *equilibrium equation* (E), and which is reminescent - although different from - of the Hamilton-Jacobi-Bellman (HJB) equation of optimal control. Note that there is also an (HJB) equation associated with problem (1),(2), but it is different from the equilibrium equation, and characterizes optimal strategies instead of equilibrium ones. We will see that the only case when equations (E) and (HJB) coincide is the case of exponential discount, and then equilibrium strategies are also optimal strategies.

The equilibrium equation comes in two different guises: an integrated form (IE) and a differentiated form (DE). We first derive the integrated form, and then we show that it is equivalent to the differentiated one. Finally, under suitable technical conditions on the utility function u and the function f, we show that solutions to the equilibrium equation exist close to the terminal time T.

Given a strategy $\sigma(t, k)$, we shall be dealing with the differential equation:

$$\frac{dk(s)}{ds} = f(s, k(s)) - \sigma(s, k(s)) \tag{13}$$
$$k(t) = k$$

We shall denote by $\mathcal{K}(s, t, k)$ the flow associated with this equation, that is the value at time s of the solution of (13) which takes the value k at time t. It is defined by:

$$\frac{\partial \mathcal{K}(s, t, k)}{\partial s} = f(s, \mathcal{K}(s, t, k)) - \sigma(s, \mathcal{K}(s, t, k)) \tag{14}$$
$$\mathcal{K}(t, t, k) = k. \tag{15}$$

In other words, $\mathcal{K}(s,t,k)$ is the value at time s of the solution of:

$$\frac{dk}{ds} = f(s,k) - \sigma(s,k) \tag{16}$$

which takes the value k at time t.

3.1 Equilibrium characterization

We shall say that a function $V : [0,T] \times R^d \to R$ satisfies *the integrated equilibrium equation* (IE) if we have, for every $t \in [0,T]$ and every k :

$$V(t,k) = \int_t^T h(s-t) u \circ i\left(\frac{\partial V}{\partial k}(s, k_0(s))\right) ds + h(T-t) g(k_0(T)) \qquad \text{((IE))}$$

where:

$$\begin{aligned} \frac{dk_0}{ds} &= f(s, k_0(s)) - i \circ \frac{\partial V}{\partial k}(s, k_0(s)) \\ k_0(t) &= k \end{aligned}$$

Note that every solution of (IE) must satisfy the *boundary condition*:

$$V(T,k) = g(k) \quad \forall k \tag{BC}$$

The following theorem characterizes the equilibrium strategies and its proof is given in the Appendix A.

Theorem 2 *Let $\sigma : [0,T] \times R^d \to R^d$ be jointly continuous, and continously differentiable with respect to k and let $\mathcal{K}$ be the associated flow defined by (14), (15). Suppose σ is an equilibrium strategy for the intertemporal decision model (3),(4). Then the function:*

$$V(t,k) = \int_t^T h(s-t) u(\sigma(s, \mathcal{K}(s,t,k))) ds + h(T-t) g(\mathcal{K}(T,t,k)), \tag{17}$$

satisfies the integrated equilibrium equation (IE) and we have:

$$\frac{\partial u}{\partial c}(\sigma(t,k)) = \frac{\partial V}{\partial k}(t,k) \tag{18}$$

Conversely, if a function V is twice continuously differentiable and satisfies the integrated equilibrium equation (IE), then:

$$\sigma(t,k) = i\left(\frac{\partial V}{\partial k}(t,k)\right)$$

is an equilibrium strategy.

Relation (18) says that, along an equilibrium path, the effect of an increment to current wealth on future utility, $\frac{\partial V}{\partial k}(t,k)$, must balance the effect of an increment to current consumption on current utility, $\frac{\partial u}{\partial c}(\sigma(t,k))$. Thus, relation (18) reflects the usual tradeoff between utility derived from current consumption and utility value of saving.

From now on, we rewrite (IE) in the form

$$V(t,k) = \int_t^T h(s-t)u\left(\sigma(s, \mathcal{K}(s,t,k))\right) ds + h(T-t)g\left(\mathcal{K}(T,t,k)\right), \tag{19}$$

with the understanding that $\mathcal{K}(s,t,k)$ is the flow associated with $\sigma(t,k) = i \circ \frac{\partial V}{\partial k}(t,k)$.

The following proposition gives a differentiated version of the equilibrium equation.

Proposition 3 *Assume that a function $V(t,k)$ is twice continuously differentiable. Then V satisfies the integrated equilibrium equation (IE) if and only if it satisfies the differentiated equilibrium equation:*

$$\begin{aligned}\frac{\partial V}{\partial t}(t,k) + \int_t^T h'(s-t)u\left(\sigma(s,\mathcal{K}(s,t,k))\right) ds + h'(T-t)g(\mathcal{K}(T,t,k)) \\ + \tilde{u}\left(\frac{\partial V}{\partial k}(t,k)\right) + \frac{\partial V}{\partial k}(t,k)f(t,k) = 0,\end{aligned} \tag{DE}$$

for all $(t,k) \in [0,T] \times R^d$, with the boundary condition

$$V(T,k) = g(k). \tag{BC}$$

Appendix B proves this proposition. It may be useful to rewrite it in the following way:

$$\rho(t,k) = \frac{1}{V}\left(u(\sigma(t,k)) + \frac{\partial V}{\partial t}(t,k) + \frac{\partial V}{\partial k}(t,k)\frac{\partial \mathcal{K}}{\partial s}(t,t,k)\right) \tag{20}$$

where:

$$\rho(t,k) = -\frac{\int_t^T \frac{h'(s-t)}{h(s-t)}h(s-t)u\left(\sigma(s,\mathcal{K}(s,t,k))\right) ds + \frac{h'(T-t)}{h(T-t)}h(T-t)g\left(\mathcal{K}(T,t,k)\right)}{\int_t^T h(s-t)u\left(\sigma(s,\mathcal{K}(s,t,k))\right) ds + h(T-t)g\left(\mathcal{K}(T,t,k)\right)}$$

is interpreted as an effective discount rate. Equation (20) then tells us that, along an equilibrium path, the relative changes in value to the consumer must be equal to the effective discount rate.

Finally, when the discount rate is exponential, the effective discount rate is just the constant discount rate $\rho = -h'(t)/h(t)$ and equation (DE) becomes simply the familiar (HJB) equation.

Corollary 4 *With the exponential discounting $h(s) = e^{-\rho s}$, the (DE) equation reduces to:*

$$\frac{\partial V}{\partial t}(t,k) - \rho V(t,k) + \tilde{u}\left(\frac{\partial V}{\partial k}(t,k)\right) + \frac{\partial V}{\partial k}(t,k)f(t,k) = 0 \tag{21}$$

Proof. In the exponential case, equations (IE) and (DE) become:

$$V(t,k) = \int_t^T e^{-\rho(s-t)} u\left(\sigma(s,\mathcal{K}\left(s,t,k\right))\right) ds + e^{-\rho(T-t)} g\left(\mathcal{K}\left(T,t,k\right)\right),$$

$$\frac{\partial V}{\partial t}(t,k) - \rho \int_t^T e^{-\rho(s-t)} u\left(\sigma(s,\mathcal{K}\left(s,t,k\right))\right) ds - \rho e^{-\rho(T-t)} g(\mathcal{K}\left(T,t,k\right)) \\ + \tilde{u}\left(\frac{\partial V}{\partial k}(t,k))\right) + \frac{\partial V}{\partial k}(t,k) f(t,k) = 0.$$

Comparing, we immediately get (21). ■

3.2 Existence

Neither equation (IE) nor equation (DE) are of a classical mathematical type. If it were not for the integral term, equation (DE) would be a first-order partial differential equation of known type (Hamilton-Jacobi), but this additional term (an integral along the trajectory of the flow (14) associated with the solution $V\left(t,k\right)$ creates additional complications.

In the sequel, we will solve that equation explicitly in particular cases. The questions of existence and uniqueness in the general case are very much open. In forthcoming work, we provide a local existence result:

Theorem 5 *Assume that all data (u, f, g and h) are analytic functions. Then, for every $\bar{k}$, there are numbers $\varepsilon > 0, \eta > 0$ and a function $V\left(t,k\right)$, defined for $T - \varepsilon \leq t \leq T$ and $\left\|k - \bar{k}\right\| \leq \eta$, such that V satisfies (DE) and (BC)*

Recall that a function is analytic at a given point if its Taylor expansion at that point has a non-zero radius of convergence. It is analytic if it is analytic at every point. The proof of the theorem relies on a generalized version of the classical Cauchy-Kowalewska theorem due to Nishida and Nirenberg [11].

3.3 The infinite-horizon problem

In the sequel, we will be looking at the infinite-horizon problem, whereby the benefit to the decision-maker at time t of a future consumption path $s \to c\left(s\right), s \geq t$, is:

$$\int_t^\infty h\left(s-t\right) u\left(c\left(s\right)\right) ds$$

The change of variables $s' = s - t \geq 0$ brings that integral to the form:

$$\int_0^\infty h\left(s'\right) u\left(c\left(s'+t\right)\right) ds'$$

which is the benefit which the decision-maker at time 0 derives from a future consumption path $s' \to c\left(s'+t\right)$.

Assume now that the problem is stationary, meaning that the production function $f(t,k)$ does not depend on t :

$$f(t,k) = f(k)$$

In that case, if the decision-maker at time t resets his watch, so that time s becomes $s-t$, she faces exactly the same problem as the decision-maker at time 0. Under these circumstances, it is natural to expect that, if both decision-makers have the same capital k, they will get the same equilibrium value:

$$V(t,k) = V(k) \quad \forall k$$

We will now look directly for time-independent value functions. Consider the equations:

$$V(k) = \int_0^\infty h(t)\, u\left(i \circ \frac{\partial V}{\partial k}\left(\mathcal{K}(t,0,k)\right)\right) dt \tag{22}$$

$$0 = \frac{\partial V}{\partial k}(k) f(k) + \int_0^\infty h'(t) u \circ i \left(\frac{\partial V}{\partial k}(\mathcal{K}(t,0,k)\right) dt + \tilde{u}\left(\frac{\partial V}{\partial k}(k)\right) \tag{23}$$

Lemma 6 *If a C^2 function $V(k)$ satisfies equation (22) or (23), then $V(t,k) :=$ $V(k)$ is a value function for the infinite-horizon problem.*

Proof. It is enough to show it for equation (22). We have to prove that, for every t, we have:

$$V(k) = \int_t^\infty h(s-t)\, u\left(i \circ \frac{\partial V}{\partial k}\left(\mathcal{K}(s-t,t,k)\right)\right) ds$$

Note that the differential equation (14) becomes autonomous, and the function $\mathcal{K}$ displays the additional property

$$\mathcal{K}(s,t_1,k) = \mathcal{K}(s-t_2, t_1-t_2, k), \quad 0 < t_2 < t_1 < s.$$

Changing variables in the integral, we get:

$$V(k) = \int_0^\infty h(s)\, u\left(i \circ \frac{\partial V}{\partial k}\left(\mathcal{K}(s,0,k)\right)\right) ds$$

which is precisely equation (22). ■

If $V(k)$ satisfies (22) or (23), the corresponding equilibrium strategy:

$$\sigma(k) = i \circ \frac{\partial V}{\partial k}$$

which is time-independent, will be called *stationary.* Note that a stationary problem may have non-stationary equilibria.

4 The consumption-saving problem

In this section, we assume $d = 1$, so that there is only one good, and we will be looking at a special case of the infinite-horizon problem. At any point in time $s \in [0, \infty)$ the consumer has a stock of wealth $k(s) \in (0, \infty)$ and receives a flow of labor income $w(s)$ as well as a flow of interest income $r(s)k(s)$. Beginning with a capital stock $k \in (0, \infty)$ at time t, we formulate the consumption-saving problem by

$$\max \int_t^{\infty} h(s-t)\, u(c(s))\, ds \tag{24}$$

$$\frac{dk(s)}{ds} = w(s) + r(s)k(s) - c(s), \;\; k(t) = k_t. \tag{25}$$

This is a special case of the general problem (3), (4), with:

$$f(t,k) := w(t) + r(t)\, k$$

We emphasize that at any point in time $t \in [0, \infty)$ the consumer takes as given the interest rate $r(t)$ and the wage $w(t)$. Since $r(t)$ and $w(t)$ are time-dependent, we expect the value function to be non-stationary, even though the horizon is infinite.

Equation (IE) becomes

$$V(t,k) = \int_t^{\infty} h(s-t) u \circ i \left(\frac{\partial V}{\partial k}(s, \mathcal{K}(s,t,k)) \right) ds \tag{26}$$

where the flow $\mathcal{K}(s,t,k)$ solves

$$\frac{d\mathcal{K}(s,t,k)}{ds} = w(s) + r(s)\mathcal{K}(s,t,k) - i \circ \frac{\partial V}{\partial k}(s, \mathcal{K}(s,t,k)) \tag{27}$$

$$\mathcal{K}(t,t,k) = k. \tag{28}$$

The next subsection gives explicit solutions when the utility function is in the CRRA class.

4.1 CRRA preferences

In this section, we shall assume that $d = 1$ and that the utility function takes one of the forms:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0$$

$$u(c) = \ln c$$

the latter corresponding to $\gamma = 1$. An explicit construction of the equilibrium strategy will be shown to be possible under an additional assumption:

Assumption A**:** *There is at least one non-negative function $t \to \lambda(t)$ which solves the fixed-point problem:*

$$\lambda(t) = \left[\int_t^\infty \lambda(s)^{1-\gamma} \exp\left[-(1-\gamma)\int_t^s (\lambda(u) - r(u))du\right] h(s-t)ds\right]^{-\frac{1}{\gamma}} \quad (29)$$

Proposition 7 *If the utility function is CRRA and $\lambda(t)$ is given by (29), the strategy*

$$\sigma(t,k) = \lambda(t)\left[k + \int_t^\infty \exp\left[-\int_t^s r(u)du\right] w(s)ds\right] \quad (30)$$

is an equilibrium strategy for the infinite-horizon problem. The associated value function is given by

$$V(t,k) = [\lambda(t)]^{-\gamma} \frac{\left[k + \int_t^\infty \exp\left[-\int_t^s r(u)du\right] w(s)ds\right]^{1-\gamma}}{1-\gamma}.$$

The equilibrium policy (30) consists of consuming the proportion $\lambda(t)$ of current wealth; the latter is the sum of the current capital stock and the present value of future wages.

The proof of Proposition 7 is given in Appendix C

We now investigate equation (29) more closely. There are three cases where it can be solved easily:

4.1.1 Constant discounting

In the case when $h(s) = e^{-\rho s}$, we find that the function

$$\bar{\lambda}(t) := \frac{\exp\left[\frac{1}{\gamma}\int_0^t ((1-\gamma)r(s) - \rho)\, ds\right]}{\int_t^\infty \exp\left[\frac{1}{\gamma}\int_0^s ((1-\gamma)r(u) - \rho)\, du\right] ds}$$

solves the equation (29) provided that the above integrals are well-defined. Therefore, the policy

$$\sigma(t,k) = \bar{\lambda}(t)\left[k + \int_t^\infty \exp\left[-\int_t^s r(u)du\right] w(s)ds\right] \quad (31)$$

is an equilibrium policy. Note that this is precisely the optimal policy from the time t perspective, which was expected anyway, since, with exponential discount, optimal policies are equilibrium policies.

4.1.2 Logarithmic utility

We are now back with a general discount function $h(t)$, but we choose a particular utility function, namely $u(c) = \ln c$, so that $\gamma = 1$. The equation reduces to:

$$\lambda(t) = \left[\int_t^\infty h(s-t)ds\right]^{-1} = \left[\int_0^\infty h(t)\, dt\right]^{-1}$$

So λ is constant (in spite of the fact that the interest rate on capital $r(t)$ is time-dependent). This fact was first observed by Barro [4]. The corresponding equilibrium strategy is:

$$\sigma(t,k) = \frac{1}{\int_0^\infty h(s)ds} \left[k + \int_t^\infty \exp\left[-\int_t^s r(u)du \right] w(s)ds \right] \tag{32}$$

4.1.3 Knife-edge case

Assume the interest rate on capital is constant and given by

$$r = \left(\int_0^\infty h(s)ds \right)^{-1}$$

We seek to solve equation (29) for a constant λ. This yields:

$$\begin{aligned}
\lambda &= \left[\int_t^\infty \lambda^{1-\gamma} e^{-(1-\gamma)(\lambda-r)(s-t)} h(s-t)ds \right]^{-\frac{1}{\gamma}} \\
&= \left[\int_t^\infty \lambda^{1-\gamma} h(s-t)ds \right]^{-\frac{1}{\gamma}} = \lambda^{-\frac{1-\gamma}{\gamma}} \left[\int_0^\infty h(s)ds \right]^{-\frac{1}{\gamma}}
\end{aligned}$$

so that $\lambda = r$. The corresponding equilibrium is given by:

$$\sigma(t,k) = r \left[k + \int_t^\infty e^{-r(s-t)} w(s)ds \right].$$

Given that along this equilibrium path, the consumers will consume the annuity value of the wealth, the above equilibrium is consistent with Friedman's permanent-income model. Note that this equilibrium strategy would also be the optimal strategy for the case of a constant discount rate equal to r, so that $h(s) = e^{-rs}$.

4.2 Constant interest rate

In this subsection, we shall assume that the interest rate on capital is constant:

$$r(t) = r$$

The following examples provide, for some specific discount functions h, explicit formulas for some equilibrium strategies with constant propensity to consume out of wealth.

4.2.1 Exponential discount

When the discount function is exponential $h(s) = e^{-\rho s}$, the equation (29) for λ takes the form

$$1 = \lambda \int_t^\infty e^{-(\rho + (\lambda - r)(1-\gamma))(s-t)} ds,$$

where $r > 0$ is the constant interest rate. If the above integral is well defined, we see that:

$$\lambda = r + \frac{\rho - r}{\gamma} =: \lambda_0.$$

The policy

$$\sigma(t,k) = \lambda_0 \left[k + \int_t^\infty e^{-r(s-t)} w(s) ds \right]$$

is an equilibrium policy provided $\lambda_0 > 0$, that is $\rho - r(1-\gamma) > 0$.

4.2.2 A mixture of exponential discount functions

For convenience, assume that $\gamma > 1$.

Consider the case when the discount function is the mixture of two exponential functions, that is:

$$h(s) = \omega e^{-\rho_1 s} + (1-\omega) e^{-\rho_2 s},$$

where $0 < \rho_1 < \rho_2$, and $\omega \in [0,1]$. The instantaneous discount rate associated to h at time t is

$$-\frac{h'(t)}{h(t)} = \rho_1 + (\rho_2 - \rho_1) \frac{1-\omega}{\omega e^{(\rho_2 - \rho_1)t} + (1-\omega)}$$

and is gradually declining from $\rho_0 := \omega\rho_1 + (1-\omega)\rho_2$ (at time $t = 0$) to ρ_1 (at time $t = \infty$). Therefore, this specification captures the idea that discount rates decline with the horizon over which utility is discounted, a feature that O'Donoghue an Rabin ([18], [19]) call the "present bias". The mixture of exponential discount function also corresponds to what Harris and Laibson [10] call the "auxiliary model".

If the discount rate were constant and equal to the long term value ρ_1, we would have a stationary equilibrium policy where $\underline{\lambda} = r + (\rho_1 - r)/\gamma$.If the discount rate were constant and equal to the short term value ρ_0, we would have a stationary equilibrium policy where $\overline{\lambda} = r + (\rho_0 - r)/\gamma$. Each of them would be optimal in its own context, given that the discount rate is constant.

In the general case where $\omega \in (0,1)$, so that the discount rate declines from ρ_0 to ρ_1, we look for an equilibrium policy where the propensity to consume out of wealth is a constant λ. After integrating, equation (29) turns out to be equivalent to the following:

$$f(\lambda) := \frac{\omega}{\rho_1 + (\lambda - r)(1-\gamma)} + \frac{1-\omega}{\rho_2 + (\lambda - r)(1-\gamma)} - \frac{1}{\lambda} = 0, \tag{33}$$

provided that the integrability conditions:

$$\rho_i + (\lambda - r)(1-\gamma) > 0, \quad i = 1,2 \tag{34}$$

are satisfied.

The function f is increasing on the interval $(0, r + \frac{\rho_1}{\gamma - 1})$ and furthermore $f(0) = -\infty$ and $f(r + \frac{\rho_1}{\gamma-1}) = +\infty$. Therefore there must exist a unique value $\lambda_1 \in (0, r + \frac{\rho_1}{\gamma-1})$ such that $f(\lambda_1) = 0$. Recalling that $\gamma > 1$, it is easy to see that λ_1 satisfies the integrability condition (34) and a further inspection reveals that λ_1 is the unique solution of the recursion (33) satisfying the integrability condition (34). Therefore, λ_1 gives rise to an equilibrium strategy.

Evaluating f at $\underline{\lambda}$ gives

$$f(\underline{\lambda}) = \gamma(1-\gamma)\left[\frac{1}{\rho_1 + \gamma(\rho_2 - \rho_1) - r(1-\gamma)} - \frac{1}{\rho_1 - r(1-\gamma)}\right] < 0,$$

and since f is increasing, we obtain that $\underline{\lambda} < \lambda_1$.

If the interest rate r has the precise value:

$$r = \frac{1}{\frac{\omega}{\rho_1} + \frac{1-\omega}{\rho_2}}$$

then $\lambda_1 = r$ is the solution, and in that case $\underline{\lambda} < \lambda_1 < \bar{\lambda}$.

4.2.3 Quasi hyperbolic discount.

We define the discount function (in continuous time) as

$$h(s) = \begin{cases} e^{-\rho s} & \text{for } 0 \leq s \leq \tau \\ \delta e^{-\rho s} & \text{for } s > \tau \end{cases}$$

where $\tau > 0$ and $\delta \in (0, 1]$.

When $\delta = 1$, the discount is exponential and the equilibrium is the one described in the preceding subsection assuming that $\tilde{\rho} = \rho - r(1-\gamma) > 0$. If $\delta < 1$, assuming a time invariant propensity to consume out of wealth, and integrating the equation (29) yields:

$$1 = \frac{\lambda}{\tilde{\rho} + \lambda(1-\gamma)}\left[1 - (1-\delta)e^{-(\tilde{\rho}+\lambda(1-\gamma))\tau}\right],$$

or equivalently $f(\lambda) = 0$, where:

$$f(\lambda) = \gamma - (1-\delta)e^{-(\tilde{\rho}+\lambda(1-\gamma))\tau} - \frac{\tilde{\rho}}{\lambda}$$

provided that an integrability condition holds:

$$\tilde{\rho} + \lambda(1-\gamma) > 0 \tag{35}$$

If $0 < \gamma < 1$, we see that $f' > 0$ and thus f is non decreasing. On the other hand, when $f(\lambda) \to -\infty$ when $\lambda \to 0, \lambda \geq 0$, and $f(\lambda) \to \gamma$ when $\lambda \to \infty$ so that there must exist a unique λ_2 such that $f(\lambda_2) = 0$. Since $\gamma < 1$, the integrability condition (35) is satisfied

If $\gamma > 1$, we have $f(\lambda) \to -\infty$ when $\lambda \to 0, \lambda > 0$, and $f\left(-\frac{\tilde{\rho}}{1-\gamma}\right) = \delta > 0$, so that f has at least one root λ_2 satisfying the integrability condition.

So the existence of an equilibrium strategy with a constant propensity to consume $\lambda(t) = \lambda_2$ is proved in all cases.

In the limiting case when $\tau \to 0$, whe obtain the instant gratification model of Harris and Laibson [10]. Then:

$$\lambda_2 \to \frac{\rho + r(\gamma - 1)}{\delta + \gamma - 1}$$

which satisfies the integrability condition (35) when $\gamma > 1$.

4.2.4 General hyperbolic discount function

We consider the discount function

$$h(s) = \frac{1}{(1+\alpha s)^{\frac{\beta}{\alpha}}} e^{-\rho s}, \quad \alpha > 0, \beta > 0 \text{ and } \rho > 0$$

specified by Luttmer and Mariotti [17] and which particularizes, when $\rho = 0$, the generalized hyperbolic discount function reported in Loewenstein and Prelec [16]. The resulting discount rate:

$$-\frac{h'(s)}{h(s)} = \rho + \frac{\beta}{1+\alpha s}$$

is smoothly declining from $\rho + \beta$ (at $s = 0$) to ρ (at $s = \infty$). The coefficient α determines how close the discount function h is to the exponentials $e^{-\rho s}$ and $e^{-(\rho+\beta)s}$.

The equation (29) becomes

$$1 = \lambda \int_0^\infty \frac{1}{(1+\alpha s)^{\frac{\beta}{\alpha}}} e^{-(\rho + (\lambda - r)(1-\gamma))s} ds$$

provided the integrability condition $\rho + (\lambda - r)(1-\gamma) > 0$ is satisfied.

We define the function

$$f(\lambda) = \int_0^\infty \frac{1}{(1+\alpha s)^{\frac{\beta}{\alpha}}} e^{-(\rho + (\lambda - r)(1-\gamma))s} ds - \frac{1}{\lambda}$$

and verify that $f(0) = -\infty$, $f\left(r + \frac{\rho}{\gamma-1}\right) = +\infty$ and $f' > 0$. Therefore, there exist a unique

$$\lambda_3 \in (0, r + \frac{\rho}{\gamma - 1})$$

such that $f(\lambda_3) = 0$ and such that the integrability condition $\rho + (\lambda_3 - r)(1-\gamma) > 0$ is satisfied.

4.3 Comparative analysis.

We want compare the equilibrium strategy with the strategy which, from the point of view of the decision-maker at time $t = 0$, is optimal. We shall do so in the case when the interest rate r and the wage w are constant, and when $u(c) = \ln c$.

The equilibrium strategy, as we saw earlier, then is time-independent and consists of consuming a constant fraction of current wealth:

$$\sigma(k) = \frac{1}{\int_0^\infty h(s)ds}\left[k + \frac{w}{r}\right] \tag{36}$$

Note that, for the model to be meaningful, we must have:

$$r > \frac{1}{\int_0^\infty h(s)ds}$$

otherwise equation (36) would mean that in equilibrium, consumption is greater that income. This makes sense: if the interest on capital is lower than the psychological discount rate, there is no point in investing.

Let us put ourselves in the shoes of the decision-maker at time $t = 0$, endowed with a capital k_0, and find the optimal strategy from her point of view. Solving the optimal control problem:

$$\max \int_0^\infty h(t) \ln c(t)\, dt$$
$$\frac{dk}{dt} = rk + w - c, \quad k(0) = k_0$$

we find, by the Euler-Lagrange equation:

$$\frac{1}{c}\frac{dc}{dt} = r + \frac{h'(t)}{h(t)}$$

which we integrate, to get:

$$c(t) = c_0 h(t) e^{rt}$$

Substituting into the dynamics, we get:

$$\frac{dk}{dt} = rk + w - c_0 h(t) e^{rt}$$

which we integrate, to get:

$$k(t) = \left(k_0 + \frac{w}{r} - c_0 \int_0^t h(s)\, ds\right) e^{rt} - \frac{w}{r}$$

Because of the transversality condition at infinity, we must have:

$$c_0 = \frac{1}{\int_0^\infty h(s)\, ds}\left(k_0 + \frac{w}{r}\right) \tag{37}$$

and the optimal propensity to consume at time t is:

$$\begin{aligned}\lambda(t) &= \frac{c(t)}{k(t)+\frac{w}{r}} = \frac{c_0 h(t)}{\left(k_0+\frac{w}{r}-c_0\int_0^t h(s)\,ds\right)} \\ &= \frac{h(t)}{\left(H-\int_0^t h(s)\,ds\right)} = h(t)\left(\int_t^\infty h(s)\,ds\right)^{-1}\end{aligned}$$

At time $t=0$, we find $\lambda(0) = \left(\int_0^\infty h(s)\,ds\right)^{-1}$. This is precisely the equilibrium value, as defined by (36). The optimal propensity to consume, $\lambda(t)$, is time-dependent, and deviates from the equilibrium value $\lambda(0)$. Note that:

$$\lambda'(0) = \frac{1}{\int_0^\infty h(s)\,ds}\left(h'(0)+\frac{1}{\int_0^\infty h(s)\,ds}\right)$$

so that $\lambda(t)$ may be greater or smaller than the equilibrium value, according to the characteristics of the discount function h. Applying the optimal strategy (from the point of view of time 0) yields the following dynamic (we set $\int_0^\infty h(s)\,ds = H$ for the sake of convenience) ::

$$\begin{aligned}k(t) &= \left(k_0+\frac{w}{r}-\left(k_0+\frac{w}{r}\right)\frac{1}{H}\int_0^t h(s)\,ds\right)e^{rt}-\frac{w}{r} \\ c(t) &= c_0 h(t)\,e^{rt} = \frac{1}{H}\left(k_0+\frac{w}{r}\right)h(t)\,e^{rt}\end{aligned}$$

Applying the equilibrium strategy yields the following dynamics:

$$\begin{aligned}k(t) &= k_0 e^{(r-1/H)t} + w\frac{1-1/rH}{r-1/H}\left(1-e^{(r-1/H)t}\right) \\ c(t) &= \frac{1}{H}\left(\frac{w}{r}+k_0 e^{(r-1/H)t}+w\frac{1-1/rH}{r-1/H}\left(1-e^{(r-1/H)t}\right)\right)\end{aligned}$$

Note, however, a remarkable fact. Define the *naive strategy* as follows: every decision-maker acts as if she could commit her successors; she computes the control $c(s)$ which is optimal on the interval $[t,\infty]$, and consumes $c(t)$. From the previous analysis it follows that the naive strategy is an equilibrium strategy. This, of course, is particular to the logarithmic case $u(c)=\ln c$.

5 Indeterminacy in the Ramsey growth problem

We now go back to the general problem (3), (4) in the stationary case, where the production function is given by:

$$f(t,k) = f(k)$$

We then interpret the problem as the Ramsey problem in growth theory. It is well-known, and described for instance in the textbook by Barro and Sala-i-Martin [5], that there are two versions to that problem:

1. *The centralized version.* A benevolent planner, seeking to maximize the integral (3) which measures global welfare, determines an optimal growth strategy $\sigma(t,k)$ and commits citizens to consume $\sigma(t,k)$ and to invest the remainder

2. *The decentralized version.* Future interest rates $r_1(t)$ and future wages $w_1(t)$ are common knowledge. The representative individual then solves the consumption-saving problem, taking $r_1(t)$ and $w_1(t)$ as given. This determines the rate of investment at any time t. This is turn determines the wages $w_2(t)$ that the production sector can offer, and the capital rental interest $r_2(t)$. One wants $r_1 = r_2$ and $w_1 = w_2$.

In the case of exponential discount, $h(t) = e^{-\rho t}$, both problems have the same solution (see [5]): if $\sigma(t,k)$ is a solution of the centralized problem, and $k(t)$ the corresponding optimal trajectory, then $w_1(t) := f(k) - k(t) f'(k(t))$ and $r_1(t) := f'(k)$ have the property that $r_1 = r_2$ and $w_1 = w_2$.

These notions naturally extend to more general discount functions. In the absence of commitment technology, one must replace optimal policies by equilibrium policies, and one is naturally led to two notions of and equilibrium growth policy:

1. *The centralized version.* There is a succession of benevolent planners, each of them holding power during an infinitesimal period of time, and having the ability to commit their contemporaries in the consumption and production sectors during that period. They agree on an equilibrium strategy $\sigma(t,k)$ for the problem (3), (4).

2. *The decentralized version.* Future interest rates $r_1(t)$ and future wages $w_1(t)$ are common knowledge. There is a succession of representative individuals, and they agree on an equilibrium strategy for the consumption-saving problem (24), (25), taking $r_1(t)$ and $w_1(t)$ as given. This determines the rate of investment at any time t. This is turn determines the wages $w_2(t)$ that the production sector can offer, and the capital rental interest $r_2(t)$. One wants $r_1 = r_2$ and $w_1 = w_2$.

In continuity with the results for the exponential discount, one would naturally expect that the two problems coincide, but this is no longer the case.

The results in Barro [4] pertain to the second problem. To the best of our knowledge the first one, that is, the study of the planner's problem in optimal growth theory under time inconsistency, has not been studied. The remainder of this paper is devoted to shedding some light on that problem. As in classical growth theory, we will concentrate on the one-dimensional case: $d = 1$.

Definition 8 *Take a point $\bar{k}$. We shall say that $\bar{k}$ is an* equilibrium point *if there is a stationary equilibrium strategy $\sigma(k)$, defined on a neighbourhood Ω of $\bar{k}$ in R^d, and such that all trajectories of (13) starting inside Ω when $t = 0$ converge to $\bar{k}$ when $t \to \infty$.*

It follows from the definition that the trajectory starting from $\bar{k}$ is $\bar{k}$ itself: the solution of (13) with $k(0) = \bar{k}$ is $k(t) = \bar{k}$ for all t. Denoting by $\bar{c}$ the consumption along that trajectory, we must have:

$$\bar{c} = f\left(\bar{k}\right)$$

Theorem 9 *Assume that $\bar{k}$ is an equilibrium point, and that the corresponding value function $V(k)$ is C^2 in a neighbourhood of $\bar{k}$. Then the number α defined by:*

$$\alpha := \frac{V''\left(\bar{k}\right)}{u''\left(\bar{c}\right)}$$

must satisfy:

$$\alpha \geq f'\left(\bar{k}\right) > 0 \tag{38}$$

If $\alpha > f'\left(\bar{k}\right)$, then:

$$\alpha \int_0^\infty h(t) \exp\left[\left(f'\left(\bar{k}\right) - \alpha\right) t\right] dt = 1 \tag{39}$$

The proof is given in Appendix D

Corollary 10 *Set $h(t) = e^{-\rho t}$. Assume that $\bar{k}$ is an equilibrium point. Then:*

$$f'\left(\bar{k}\right) = \rho \tag{40}$$

and,

$$\alpha \equiv \frac{V''(\bar{k})}{u''(\bar{c})} = f'(\bar{k}) \left[\frac{1 + \sqrt{1 + 4\frac{u'(\bar{k})f''(\bar{k})}{\rho^2 u''(\bar{k})}}}{2}\right] > f'(\bar{k}) \tag{41}$$

In the exponential case equation (39) degenerates: it sets no condition on α, but determines $\bar{k}$ through (40). This is the well-known relation for the optimal growth path, which usually is obtained by the transversality condition at infinity, and which here is derived in a novel way.

In the general case, as we will see in the following example, equation (39) does not determine $\bar{k}$: it determines α as a function of $\bar{k}$. The proof is given in the appendix.

Proposition 11 *Set $h(t) = e^{-rt}$ on $[0,\ T]$, and $h(t) = 0$ for $t > T$. Assume that $\bar{k}$ is an equilibrium point. Then, there exists a decreasing function $\varphi :]0, \infty[\to]0, \infty[$, with:*

$$\begin{aligned} &\varphi(\alpha) \to \infty \text{ when } \alpha \to 0 \\ &\varphi(1) = 1 \end{aligned}$$

and a number $a(T) \leq 1/T$ such that conditions (39), (38) are equivalent to the following:

$$0 < f'\left(\bar{k}\right) - r \leq 1/T \tag{42}$$

$$\alpha T = \varphi\left(\left[f'\left(\bar{k}\right) - r\right] T\right) \tag{43}$$

In other words, there is a continuous family of solutions to equation (39), one for each $\bar{k}$ such that $f'\left(\bar{k}\right)$ falls in the interval $(\rho,\ \rho+1/T)$. The corresponding $\alpha\left(\bar{k}\right)$ goes to ∞ when $f'\left(\bar{k}\right)\to\rho$ and to $1/T$ when $f'\left(\bar{k}\right)\to\rho+1/T$. Note that there is no solution for $f'\left(\bar{k}\right)\leq\rho$ of $f'\left(\bar{k}\right)\geq\rho+1/T$

For future reference, we write a few properties of the function φ. They follow easily from the properties of the function xe^{-x} :

$$
\begin{aligned}
x_1 < x_2 &\Longrightarrow \varphi(x_1) < \varphi(x_2) \\
\varphi(x) &\to \infty \text{ when } x \to \infty, \\
\varphi(x) &\to 0 \text{ when } x \to \infty \\
x\varphi(x) &= \psi\left(xe^{-x}\right) \text{ with } \psi(0) = 0,\ \psi'(0) = 1, \text{when } x \geq 1
\end{aligned}
$$

We now want to know what happens when the horizon T goes to 0 (instant gratification) or ∞ (exponential discount). Let $(\alpha(T), k(T))$ be a solution of equation (39) with $h(t)$ as above:

$$
\alpha(T)\int_0^T e^{-\rho t}\exp\left[\left(f'(k(T))-\alpha(T)\right)t\right]dt = 1 \tag{44}
$$

- Let $T\to\infty$. Assume $k(T)\to\bar{k}$ and $\alpha(T)\to\bar{\alpha}>0$. Then:

$$
f'\left(\bar{k}\right)=\rho
$$

- Let $T\to 0$. Assume $k(T)\to\bar{k}$ and $\alpha(T)\to\bar{\alpha}>0$. Then:

$$
\bar{k}=0
$$

The proof of the first part follows immediately from the estimate $0 < f'(k(T))-\rho\leq 1/T$. For the second, we have $\alpha(T)=\varphi\left(f'(k(T))T-\rho T\right)/T$. Since the left-hand side converges, so must the right-hand side, so $\varphi\left(f'(k(T))T-\rho T\right)$ must go to infinity when $T\to 0$, which is only possible if $f'(k(T))\to\infty$. Since f satisfies the Inada conditions, we must have $k(T)\to 0$, as announced,

These results are conform to economic intuition. Note in particular that, when $T\to\infty$, we find again the condition $f'\left(\bar{k}\right)=\rho$ in the limit. However, for finite T, equation (39) does not determine $k(T)$, which is a striking difference with $T=\infty$. In this, as in the general case of non-constant discount, we have been unable to find any further condition that would determine $k(T)$. This would indicate non-uniqueness of possible $k(T)$, and hence a multiplicity of equilibrium strategies, one for each possible value of $k(T)$ and $\alpha(T)$. The following results, which are valid in the case of general non-constant discounts, indicates that, even with non-uniqueness, there is a definite range of possible equilibrium values for $\bar{k}$.

Corollary 12 *Assume that the discount function $h(t)$ satisfies:*

$$
e^{-\rho_2 t}\leq h(t)\leq e^{-\rho_1 t}\ \textit{for}\ t\geq 0
$$

for some $0<\rho_1\leq\rho_2$, and that $\bar{k}$ is an equilibrium point for h. Then:

$$
\rho_1\leq f'\left(\bar{k}\right)\leq\rho_2
$$

Proof. From equation (39), we have:

$$1 = \alpha \int_0^\infty h(t) \exp\left[\left(f'\left(\bar{k}\right) - \alpha\right) t\right] dt \tag{45}$$

$$\leq \alpha \int_0^\infty \exp\left[\left(-r_1 + f'\left(\bar{k}\right) - \alpha\right) t\right] dt \tag{46}$$

$$= \frac{\alpha}{r_1 - f'\left(\bar{k}\right) + \alpha} \tag{47}$$

and this gives $f'\left(\bar{k}\right) \geq r_1$. Hence the result ■

More generally, we have the following result:

Proposition 13 *Assume that the discount functions h_1 and h satisfy:*

$$h_0(t) \leq h_1(t) \quad \text{for } t \geq 0$$

Denote by K_0 and K_1 the set of equilibrium points for h_0 and h_1 respectively. Then:

$$\sup K_0 \leq \sup K_1$$

Proof. Assume otherwise, so that there exists some $\bar{k}_0 \in K_0$ such that $\bar{k}_0 > \sup K_1$. Then there exists some $\bar{k}_1 \notin K_1$ with $\bar{k}_1 < \bar{k}_0$. . Since f is strictly concave, we must have $f'\left(\bar{k}_0\right) < f'\left(\bar{k}_1\right)$. Set:

$$\varphi_0(\alpha) := \alpha \int_0^\infty h_0(t) \exp\left[\left(f'\left(\bar{k}_0\right) - \alpha\right) t\right] dt \tag{48}$$

$$\varphi_1(\alpha) := \alpha \int_0^\infty h_1(t) \exp\left[\left(f'\left(\bar{k}_1\right) - \alpha\right) t\right] dt \tag{49}$$

Since $\bar{k}_1 \notin K_1$, we have $\varphi_1(\alpha) \neq 1$ for all $\alpha > 0$. Since $\varphi_1(\alpha) \to 0$ when $\alpha \to \infty$, this implies that $\varphi_1(\alpha) < 1$ for all $\alpha > 0$. Since $h_0 \leq h_1$ and $f'\left(\bar{k}_0\right) \leq f'\left(\bar{k}_1\right)$, we have $\varphi_0 \leq \varphi_1$, such that $\varphi_1(\alpha) < 1$ for all $\alpha > 0$. ■

6 Conclusion

This paper tried to model the idea that the decision-maker at time t cannot commit her successors by imagining that she can commit her immediate successors, those in the interval $[t,\ t+\varepsilon]$, and letting $\varepsilon \to 0$. We then gave a rigourous definition of (subgame perfect) equilibrium strategies, characterize them through the equations (IE) and (DE). We give a local existence result in the analytic case.

One would, of course, like to have a global existence result, on $R^d \times [0,\ T]$, and to have weaker regularity assumptions (sufficiently differentiable instead of analytic). Unfortunately, proving such a theorem presents us with some serious mathematical challenges, and much more work is required before we understand the situation. In fact, it seems to be very similar to the situation wich prevailed on the (HJB) equation itself before the discovery of viscosity solutions by Mike Crandall and Pierre-Louis Lions when issues of existence, uniqueness and regularity where intertwined in a very unsatisfactory manner. We feel that a similar program has to be undertaken for equation (DE).

Another question is: why the continuous time ? Would it not be easier to work with discrete time, and actually get the continous case by an appropriate limiting process from the discrete case ? The answer, as we pointed out in the introduction, is that we have no existence result for subgame perfect equilibrium in the discrete case, so it is by no means clear that it is easier than the continuous case. When Aumann started the study of economies with a continuum of consumers, theorems were first proved directly, and the connection with economies with a large number of consumers came much later. For instance, the fact that if one constructs an economy with Nn agents by replicating N times an economy with n agents, the core of the large economy converges to the equilibria of the limiting economy (which has a continuum of agents) was first proved by Herbert Scarf, and was hailed as a major achievement. Here again, such limiting results may hold for equilibrium strategies, but it is another research program.

Finally, the obvious economic question is whether equilibrium strategies are observationally different from optimal strategies. The difficulty here is that, although the equilibrium strategy is defined for all (t, k), we only observe one trajectory of the dynamics, the one that starts at k_0 at time $t = 0$. Devising testable consequences for our model will be a third research program.

A Proof of Theorem 2*:*

A.1 Preliminaries

Before proceeding with the proof of the theorem, let us mention some facts about the flow $\mathcal{K}$ defined by (14), (15).

Note first that the solution of (14) which takes the value k at time t coincides with the solution of the same equation which takes the value $\mathcal{K}(s_1, t, k)$ at time s_1. In mathematical terms, this property may be stated as

$$\mathcal{K}(s_2, t, k) = \mathcal{K}(s_2, s_1, \mathcal{K}(s_1, t, k)), \quad 0 < t < s_1 < s_2 < T. \tag{50}$$

Next, consider the linearized equation around a prescribed solution $t \to k_0(t)$ of the nonlinear system (16), namely:

$$\frac{dk}{ds} = \left(\frac{\partial f}{\partial k}(s, k_0(s)) - \frac{\partial \sigma}{\partial k}(s, k_0(s)) \right) k(s) \tag{51}$$

This is a linear equation, so the flow is linear. The value at time s of the solution which takes the value k at time t is $\mathcal{R}(s, t)k$, where the matrix

$\mathcal{R}(\cdot,t) : [t,T] \longrightarrow R^{d\times d}$ satisfies:

$$\frac{d\mathcal{R}}{ds}(s,t) = \left(\frac{\partial f}{\partial k}(s,k_0(s)) - \frac{\partial \sigma}{\partial k}(s,k_0(s))\right)\mathcal{R}(s,t) \tag{52}$$

$$\mathcal{R}(t,t) = \mathbf{I} \tag{53}$$

From standard theory, it is well known that, if f and σ are C^k, then $\mathcal{K}$ is C^{k-1}, and:

$$\frac{\partial \mathcal{K}(s,t,k)}{\partial k} = \mathcal{R}(t,s)$$

$$\frac{\partial \mathcal{K}(s,t,k)}{\partial t} = -\mathcal{R}(t,s)\left(f(t,k) - \sigma(t,k)\right)$$

where $\mathcal{R}(t,s)$ is computed by setting $k_0(s) = \mathcal{K}(s,t,k)$ in formulas (52), (53).

Let us now turn to the actual proof of Theorem 2:

A.2 Necessary condition

Given an equilibrium strategy σ, define a function V by:

$$V(t,k) = \int_t^T h(s-t)u\left(\sigma(s,\mathcal{K}(s,t,k))\right)ds + h(T-t)g\left(\mathcal{K}(T,t,k)\right), \tag{54}$$

Differentiating with respect to k, we find that:

$$\begin{aligned}\frac{\partial V}{\partial k}(t,k) &= \int_t^T h(s-t)\frac{\partial u}{\partial c}\left(\sigma(s,\mathcal{K}(s,t,k))\right)\frac{\partial \sigma}{\partial k}(s,\mathcal{K}(s,t,k))\frac{\partial \mathcal{K}}{\partial k}(s,t,k)\,ds \\ &\quad + h(T-t)\frac{\partial g}{\partial k}\left(\mathcal{K}(T,t,k)\right)\frac{\partial \mathcal{K}}{\partial k}(T,t,k) \\ &= \int_t^T h(s-t)\frac{\partial u}{\partial c}\left(\sigma(s,\mathcal{K}(s,t,k))\right)\frac{\partial \sigma}{\partial k}(s,\mathcal{K}(s,t,k))\,\mathcal{R}(s,t)ds \\ &\quad + h(T-t)\frac{\partial g}{\partial k}\left(\mathcal{K}(T,t,k)\right)\mathcal{R}(T,t)\end{aligned}$$

The (IE) equation will be derived by maximizing the individual payoff $P_1(t,k,\sigma,c)$, as in formula (10). To this end, let us first notice that the function k_1 defined by (11) and (12) can be written as

$$k_1(s) = \mathcal{R}(s,t)\left(\sigma(t,k) - c\right),$$

so that the individual payoff becomes

$$\begin{aligned}P_1(t,k,\sigma,c) &= u(c) - u(\sigma(t,k)) \\ &\quad + \int_t^T h(s-t)\frac{\partial u}{\partial c}\left(\sigma(s,k_0(s))\right)\frac{\partial \sigma}{\partial k}(s,k_0(s))\,\mathcal{R}(s,t)\left(\sigma(t,k)-c\right)ds \\ &\quad + h(T-t)\frac{\partial g}{\partial k}\left(k_0(T)\right)\mathcal{R}(T,t)\left(\sigma(t,k)-c\right)\end{aligned}$$

Since u is concave and differentiable, the necessary and sufficient condition to maximize $P_1(t,k,\sigma,c)$ with respect to c is

$$\begin{aligned}\frac{\partial u}{\partial c}(c) &= \int_t^T h(s-t)\frac{\partial u}{\partial c}\left(\sigma(s,\mathcal{K}(s,t,k))\right)\frac{\partial \sigma}{\partial k}\left(s,\mathcal{K}(s,t,k)\right)\mathcal{R}(s,t)ds \\ &\quad + h(T-t)\frac{\partial g}{\partial k}\left(\mathcal{K}(T,t,k)\right)\mathcal{R}(T,t) = \frac{\partial V}{\partial k}(t,k).\end{aligned}$$

Therefore, the equilibrium strategy must satisfy

$$\frac{\partial u}{\partial c}\left(\sigma(t,k)\right) = \frac{\partial V}{\partial k}(t,k)$$

and, substituting back into equation (54), gives the (IE) equation.

A.3 Sufficient condition

Assume now that there exists a function V satisfying (IE) and (BC), and consider the strategy $\sigma = i \circ \frac{\partial V}{\partial k}$. Given any consumption choice $c \in R^d$, the payoff to the decision-maker at time t is:

$$\begin{aligned}P_1(t,k,\sigma,c) &= u(c) - u(\sigma(t,k)) \\ &\quad + \left[\int_t^T h(s-t)\frac{\partial u}{\partial c}\left(\sigma(s,k_0(s))\right)\frac{\partial \sigma}{\partial k}\left(s,k_0(s)\right)\mathcal{R}(s,t)ds\right. \\ &\quad \left. + h(T-t)\frac{\partial g}{\partial k}\left(k_0(T)\right)\mathcal{R}(T,t)\right]\left(\sigma(t,k)-c\right) \\ &= u(c) - u(\sigma(t,k)) + \frac{\partial V}{\partial k}(t,k)\left(\sigma(t,k)-c\right) \\ &= u(c) - u(\sigma(t,k)) - \frac{\partial u}{\partial c}\left(\sigma(t,k)\right)\left(c-\sigma(t,k)\right) \\ &\le 0,\end{aligned}$$

where the first equality follows from the definition of $\mathcal{R}$, the second equality is obtained by differentiating V with respect to k, the third equality follows from the definition of σ, and the last inequality is due to the concavity of u. Observing that $P_1(t,k,\sigma,\sigma(t,k)) = 0$, we see that the inequality $P_1(t,k,\sigma,c) \le 0$ proves that $c = \sigma(t,k)$ achieves the maximum so that σ is an equilibrium strategy. Q.E.D.

B Proof of Proposition 3

Let a function $V : [0,T] \times R^d \to R^d$ be given. Consider the function:

$$\varphi(t,k) = V(t,k) - \int_t^T h(s-t)u\left(\sigma(s,\mathcal{K}(s,t,k))\right)ds - h(T-t)g\left(\mathcal{K}(T,t,k)\right)$$

where $\sigma = i \circ \frac{\partial V}{\partial k}$.

Consider the value of φ along the trajectory of (13) originating from k at time t. It is given by:

$$\begin{aligned}
\psi(s,t,k) &= \varphi(s,\mathcal{K}(s,t,k)) \\
&= V(s,\mathcal{K}(s,t,k)) \\
&- \int_s^T h(x-s)u(\sigma(x,\mathcal{K}(x,s,\mathcal{K}(s,t,k))))\,dx \\
&- h(T-s)g(\mathcal{K}(T,s,\mathcal{K}(s,t,k))) \\
&= V(s,\mathcal{K}(s,t,k)) \\
&- \int_s^T h(x-s)u(\sigma(x,\mathcal{K}(x,t,k)))\,dx \\
&- h(T-s)g(\mathcal{K}(T,t,k))
\end{aligned}$$

where we have used formula (50).

We compute the derivative of this function with respect to s:

$$\begin{aligned}
\frac{\partial \psi}{\partial s}(s,k,t) = \frac{\partial V}{\partial t}(s,\mathcal{K}(s,t,k)) &+ \frac{\partial V}{\partial k}(s,\mathcal{K}(s,t,k))\left(f(s,\mathcal{K}(s,t,k)) - \sigma(s,\mathcal{K}(s,t,k))\right) \\
&+ \int_s^T h'(x-s)u(\sigma(x,\mathcal{K}(x,t,k)))\,ds + u(\sigma(s,\mathcal{K}(s,t,k)) \\
&+ h'(T-s)g(\mathcal{K}(T,t,k))
\end{aligned}$$

Since $\sigma = i \circ \frac{\partial V}{\partial k}$, we have:

$$-\frac{\partial V}{\partial k}(s,\mathcal{K}(s,t,k))\,\sigma(s,\mathcal{K}(s,t,k)) + u(\sigma(s,\mathcal{K}(s,t,k)) = \tilde{u}\left(\frac{\partial V}{\partial k}(s,\mathcal{K}(s,t,k))\right)$$

Substituting in the above, we get:

$$\begin{aligned}
\frac{\partial \psi}{\partial s}(s,k,t) &= \frac{\partial V}{\partial t}(s,\mathcal{K}(s,t,k)) + \frac{\partial V}{\partial k}(s,\mathcal{K}(s,t,k))\,f(s,\mathcal{K}(s,t,k)) \\
&+ \int_s^T h'(x-s)u(\sigma(x,\mathcal{K}(x,t,k)))\,ds + \tilde{u}\left(\frac{\partial V}{\partial k}(s,\mathcal{K}(s,t,k))\right) \\
&+ h'(T-s)g(\mathcal{K}(T,t,k))
\end{aligned}$$

If (DE) holds, then $\psi(s,k,t) = \psi(T,k,t)$, and if (BC) holds, then ψ is identically zero, so that (IE) holds. Conversely, if (IE) holds, then (BC) and (DE) obviously hold. Q.E.D.

C Proof of Proposition 7

Define the function

$$\mathcal{M}(s,t,k) = \mathcal{K}(s,t,k)e^{-\int_t^s (r(u)-\lambda(u))du}$$

with the understanding that the flow $\mathcal{K}$ is associated to the strategy (30). Then we have

$$\frac{d\mathcal{M}(s,t,k)}{ds} = w(s)e^{-\int_t^s (r(u)-\lambda(u))du} - \lambda(s)e^{-\int_t^s (r(u)-\lambda(u))du} \int_s^\infty e^{-\int_s^u r(v)dv} w(u)du$$

and after integrating this equation on $[t,s]$ we get

$$\begin{aligned}\mathcal{M}(s,t,k) = k &+ \int_t^s e^{-\int_t^u (r(x)-\lambda(x))dx} w(u)du \\ &- \int_t^s \lambda(u)e^{-\int_t^u (r(x)-\lambda(x))dx} \left[\int_u^\infty e^{-\int_u^v r(x)dx} w(v)dv\right] du. \quad (55)\end{aligned}$$

The last term of this equality may be transformed into

$$\begin{aligned}&\int_t^s \lambda(u)e^{-\int_t^u (r(x)-\lambda(x))dx} \left[\int_u^\infty e^{-\int_u^v r(x)dx} w(v)dv\right] du \\ &= \int_t^s \lambda(u)e^{\int_t^u \lambda(x)dx} \left[\int_t^\infty e^{-\int_t^v r(x)dx} w(v)dv - \int_t^u e^{-\int_t^v r(x)dx} w(v)dv\right] du \\ &= \left[\int_t^\infty e^{-\int_t^u r(x)dx} w(u)du\right] \left[\int_t^s \lambda(u)e^{\int_t^u \lambda(x)dx} du\right] - \int_t^s \lambda(u)e^{\int_t^u \lambda(x)dx} \left[\int_t^u e^{-\int_t^v r(x)dx} w(v)dv\right] du \\ &= \left[\int_t^\infty e^{-\int_t^u r(x)dx} w(u)du\right] \left[e^{\int_t^s \lambda(x)dx} - 1\right] - \int_t^s e^{-\int_t^v r(x)dx} w(v) \left[\int_v^s \lambda(u)e^{\int_t^u \lambda(x)dx} du\right] dv \\ &= \left[\int_t^\infty e^{-\int_t^u r(x)dx} w(u)du\right] \left[e^{\int_t^s \lambda(x)dx} - 1\right] - \int_t^s e^{-\int_t^v r(x)dx} w(v) \left[e^{\int_t^s \lambda(x)dx} - e^{\int_t^v \lambda(x)dx}\right] dv \\ &= \left[\int_t^\infty e^{-\int_t^u r(x)dx} w(u)du\right] \left[e^{\int_t^s \lambda(x)dx} - 1\right] - e^{\int_t^s \lambda(x)dx} \int_t^s e^{-\int_t^u r(x)dx} w(u)du \\ &\qquad + \int_t^s e^{-\int_t^u (r(x)-\lambda(x))dx} w(u)du\end{aligned}$$

where the third equality follows from Fubini Theorem. Substituting this formulation in equation (55) yields

$$\mathcal{M}(s,t,k) = k + \int_t^\infty e^{-\int_t^u r(x)dx} w(u)du - e^{\int_t^s \lambda(x)dx} \left[\int_s^\infty e^{-\int_t^u r(x)dx} w(u)du\right]$$

and therefore,

$$\mathcal{K}(s,t,k) = e^{\int_t^s (r(x)-\lambda(x))dx} \left[k + \int_t^\infty e^{-\int_t^u r(x)dx} w(u)du\right] - \int_s^\infty e^{-\int_s^u r(x)dx} w(u)du.$$

Denoting by $V(t,k)$ the utility associated to the strategy σ defined by (30), we see that

$$
\begin{aligned}
V(t,k) &= \int_t^\infty \frac{[\sigma(s,\mathcal{K}(s,t,k))]^{1-\gamma}}{1-\gamma} h(s-t)ds \\
&= \int_t^\infty \frac{\left[\lambda(s)e^{\int_t^s (r(x)-\lambda(x))dx}\right]^{1-\gamma}}{1-\gamma} \left[k + \int_t^\infty e^{-\int_t^u r(x)dx} w(u)du\right]^{1-\gamma} h(s-t)ds \\
&= \frac{\left[k + \int_t^\infty e^{-\int_t^u r(x)dx} w(u)du\right]^{1-\gamma}}{1-\gamma} \int_t^\infty \left[\lambda(s)e^{\int_t^s (r(x)-\lambda(x))dx}\right]^{1-\gamma} h(s-t)ds
\end{aligned}
$$

and that,

$$
\frac{\partial V}{\partial k}(t,k) = \left[k + \int_t^\infty e^{-\int_t^u r(x)dx} w(u)du\right]^{-\gamma} \int_t^\infty \left[\lambda(s)e^{\int_t^s (r(x)-\lambda(x))dx}\right]^{1-\gamma} h(s-t)ds.
$$

Now, since the recursion (29) is satisfied, we see that

$$
\frac{\partial V}{\partial k}(t,k) = \left[k + \int_t^\infty e^{-\int_t^u r(x)dx} w(u)du\right]^{-\gamma} [\lambda(t)]^{-\gamma} = (\sigma(t,k))^{-\gamma}.
$$

Therefore, the integrated equation (26) is satisfied which in turn establishes that the strategy σ defined by (30) is an equilibrium strategy for the non-stationary problem $(24),(25)$.

D Proof of Theorem 9

Proof. Write equation (DE) for the function $V(k)$:

$$
\tilde{u}(V') + fV' + \int_0^\infty h'(t)\, u(c(t))\, dt = 0
$$

with $c(t) = \sigma(\mathcal{K}(t,k)) = i(V'(\mathcal{K}(t,k)))$.

Differentiate it at the equilibrium point $\bar{k}$. We get:

$$
\left(\tilde{u}'\left(V'\left(\bar{k}\right)\right) + f\left(\bar{k}\right)\right) V''\left(\bar{k}\right) + f'\left(\bar{k}\right) V'\left(\bar{k}\right) + \int_0^\infty h'(t)\, u'(\bar{c})\, i'\left(V'\left(\bar{k}\right)\right) V''\left(\bar{k}\right) \frac{\partial \mathcal{K}(t,k)}{\partial k} dt \tag{56}
$$

We have $\tilde{u}'\left(V'\left(\bar{k}\right)\right) = -i\left(V'\left(\bar{k}\right)\right) = -\bar{c}$, so that the first term vanishes.

We are left with the two others. Note first that $i'(c) = -\tilde{u}''(c) = -1/u''(c)$, so that the last integral can be rewritten as follows:

$$
-\frac{u'(\bar{c})}{u''(\bar{c})} V''\left(\bar{k}\right) \int_0^\infty h'(t) \frac{\partial \mathcal{K}(t,k)}{\partial k} dt
$$

The function $y(t) = \partial \mathcal{K}(t,k)/\partial k$ is the solution of the linearized system at $\bar{k}$:

$$
\frac{dy}{dt} = \left(f'\left(\bar{k}\right) - i'\left(V'\left(\bar{k}\right)\right) V''\left(\bar{k}\right)\right) y = \left(f'\left(\bar{k}\right) - \alpha\right) y \tag{57}
$$

Since $\bar{k}$ is an attractor, the exponent $\left(f'\left(\bar{k}\right)-\alpha\right)$ must be non-positive, so $f'\left(\bar{k}\right)-\alpha\leq 0$, which is condition (38).

From now on, we assume $f'\left(\bar{k}\right)-\alpha<0$, so that the linearized equation (57) converges, and we have:

$$\frac{\partial\mathcal{K}\left(t,k\right)}{\partial k}=\exp\left[\left(f'\left(\bar{k}\right)-\alpha\right)t\right]$$

This gives us the last term in (56). We now compute the middle term by differentiating the formula for V :

$$V\left(k\right)=\int_{0}^{\infty}h\left(t\right)u\left(c\left(t\right)\right)dt$$

yielding, by the same computation:

$$\begin{aligned}V'\left(\bar{k}\right)&=-\frac{u'\left(\bar{c}\right)}{u''\left(\bar{c}\right)}V''\left(\bar{k}\right)\int_{0}^{\infty}h\left(t\right)\frac{\partial\mathcal{K}\left(t,k\right)}{\partial k}dt\\&=-\frac{u'\left(\bar{c}\right)}{u''\left(\bar{c}\right)}V''\left(\bar{k}\right)\int_{0}^{\infty}h\left(t\right)\exp\left[\left(f'\left(\bar{k}\right)-\alpha\right)t\right]dt\end{aligned}$$

Substituting in equation (56), we get:

$$\frac{u'\left(\bar{c}\right)}{u''\left(\bar{c}\right)}V''\left(\bar{k}\right)\left[f'\left(\bar{k}\right)\int_{0}^{\infty}h\left(t\right)\exp\left[\left(f'\left(\bar{k}\right)-\alpha\right)t\right]dt+\int_{0}^{\infty}h'\left(t\right)\exp\left[\left(f'\left(\bar{k}\right)-\alpha\right)t\right]dt\right]=0$$

and hence:

$$\begin{aligned}f'\left(\bar{k}\right)&=-\frac{\int_{0}^{\infty}h'\left(t\right)\exp\left[\left(f'\left(\bar{k}\right)-\alpha\right)t\right]dt}{\int_{0}^{\infty}h\left(t\right)\exp\left[\left(f'\left(\bar{k}\right)-\alpha\right)t\right]dt}\\&=\frac{\left(f'\left(\bar{k}\right)-\alpha\right)\int_{0}^{\infty}h\left(t\right)\exp\left[\left(f'\left(\bar{k}\right)-\alpha\right)t\right]dt+1}{\int_{0}^{\infty}h\left(t\right)\exp\left[\left(f'\left(\bar{k}\right)-\alpha\right)t\right]dt}\end{aligned}$$

where we have integrated by parts. This in turn gives formula (39) and concludes the proof. ■

E Proof of Corollary 10

Substitute into equation (39). We get for α the equation.

$$1=\alpha\int_{0}^{\infty}\exp\left[-\rho+f'\left(\bar{k}\right)-\alpha\right]dt=\frac{\alpha}{\rho-f'\left(\bar{k}\right)+\alpha}$$

If $f'\left(\bar{k}\right)\neq\rho$, there is no solution to this equation, so $f'\left(\bar{k}\right)$ must be equal to ρ. In order to establish (41), we differentiate twice the (DE) equation (which is here the HJB equation) and evaluate it at $\bar{k}$ and get

$$-V'(\bar{k})f''(\bar{k})=u''(\bar{c})\alpha\left(f'(\bar{k})-\alpha\right).$$

This is a quadratic equation in α and, assuming f is concave, it admits two roots,

$$\alpha = f'(\bar{k}) \left[\frac{1 \pm \sqrt{1 + 4\frac{u'(\bar{k})f''(\bar{k})}{\rho^2 u''(\bar{k})}}}{2} \right]$$

The second root is not valid because $\bar{k}$ is an attractor and hence $0 < f'(\bar{k}) \leq \alpha$. This leave us with the only possible root α given by (41). Q.E.D.

F Proof of Proposition 11

Substituting the specification of the discount function into equation (39) gives

$$1 = \alpha \int_0^T \exp \left[\left(-\rho + f'\left(\bar{k}\right) - \alpha \right) t \right] dt \tag{58}$$

Since $\alpha \geq f'\left(\bar{k}\right)$, the term $\rho - f'\left(\bar{k}\right) + \alpha$ is different from 0 and therefore the equation (58) becomes

$$\left(f'\left(\bar{k}\right) - \rho \right) T \exp \left[-T \left(f'\left(\bar{k}\right) - \rho \right) \right] = \alpha T \exp \left(-\alpha T \right) \tag{59}$$

The left inequality of (42) is simply obtained by noticing that, due to equation (58), α must be positive and the equation (59) implies then that $f'\left(\bar{k}\right) - \rho > 0$.

Now notice that equation (59) is of the type $xe^x = ye^y$. It has the obvious solution $x = y$, plus another one. The first solution gives $f'\left(\bar{k}\right) - \rho = \alpha$, contradicting the fact that $\rho - f'\left(\bar{k}\right) + \alpha \neq 0$, so it must be rejected. The second solution defines y as a function of x, say $y = \varphi(x)$ for $x > 0$, which is easily seen to be decreasing and to obey the properties given in (**??**). So $\alpha T = \varphi\left(T\left(f'\left(\bar{k}\right) - \rho\right)\right)$, and formula (43) follows.

Now, taking into account the condition $\alpha \geq f'\left(\bar{k}\right)$ given by (38), equation (43) implies

$$\varphi\left(f'\left(\bar{k}\right)T - \rho T\right) \geq f'\left(\bar{k}\right)T > f'\left(\bar{k}\right)T - \rho T.$$

Since the function φ is decreasing and $\varphi(1) = 1$, the above inequality implies

$$f'\left(\bar{k}\right)T - \rho T < 1$$

which is precisely the right inequality of of (42).

References

[1] Ainslie, George. “Specious Reward: A Behavioral Theory of Impulsiveness and Impulse Control.’ *Psychological Bull.* 82 (July 1975): 463-96.

[2] Ainslie, George. *Picoeconomics: The Interaction of Successive Motivational States within the Individual.* Cambridge: Cambridge Univ. Press, 1992.

[3] Aumann, Robert. “Markets with a Continuum of Traders.’, *Econometrica*, 32 (1964): 39-50.

[4] Barro Robert J. “Ramsey Meets Laibson in the Neoclassical Growth Model.’ *Quarterly Journal of Economics* 114 (November 1999): 1125-52.

[5] Barro Robert J., and Xavier Sala-i-Martin. “Economic Growth.’ New York: McGraw Hill, 1995.

[6] Diamond, Peter, and Kszegi Botond. “Quasi-hyperbolic discounting and early retirement.’ *Journal of Public Economics* 87 (2003): 1839-72.

[7] Frederick, Shane, Loewenstein, George, and O’Donoghue Ted. “Time Discounting and Time preference: a Critical Review.’ *Journal of Economic Literature* 40 (2002): 351-401.

[8] Harris, Christopher, and Laibson, David. “Dynamic Choices of Hyperbolic Consumers.’ *Econometrica* 69 (July 2001): 935-57.

[9] Harris, Christopher, and Laibson, David. “Hyperbolic discounting and consumption.’ eds. Mathias Dewatripont, Lars Peter Hansen, and Stephen Turnovsky, *Advances in Economics and Econometrics: Theory and Applications, Eighth World Congress of the Econometric Society,* Volume 1, 2002, pp. 258-298.

[10] Harris, Christopher, and Laibson, David. “Instantaneous Gratification.’ Working paper (2004), Harvard University.

[11] T. Kano, and T. Nishida. “ Sur les ondes de surface de l’eau avec une justification mathematique des equations des ondes en eau peu profonde.’ J. Math. Kyoto University (JMKYAZ) 19 (1979): 335-370.

[12] Karp, Larry. “Non-constant Discounting in Continuous Time.’ Working paper (2005), Berkeley, forthcoming in *Journal of Economic Theory.*

[13] Krusell, Per, and Smith, Anthony. “Consumption and Savings Decisions with Quasi-geometric Discounting.’ *Econometrica* 71(1) (2003): 365-75.

[14] Laibson, David. “Golden Eggs and Hyperbolic Discounting.’ *Quarterly Journal of Economics* 112 (May 1997): 443-77.

[15] Laibson, David. “Life-cycle Consumption and Hyperbolic Discount Functions-Evidence and an Interpretation.’ *European Economic Review* 42 (1998): 861-71.

[16] Loewenstein, George, and Prelec, Drazen. “Anomalies in Intertemporal Choice: Evidence and an Interpretation.’ *Quarterly Journal of Economics* 107 (May 1992): 573-97.

[17] Luttmer, Erzo G. J., and Mariotti, Thomas. “Subjective Discounting in an Exchange Economy.’ *Journal of Political Economy* 111 (2003): 959-89.

[18] O'Donoghue, Ted, and Rabin, Matthew. "Doing It Now or Later.' *American Economic Review* 89 (March 1999): 103-24.

[19] O'Donoghue, Ted, and Rabin, Matthew. "Choice and Procrastination.' *Quarterly Journal of Economics* 116 (February 2001): 121-60.

[20] Peleg, Bezalel, and Yaari, Menahem. "On the Existence of a Consistent Course of Actions When Tastes Are Changing.' *Review of Economic Studies* 40 (July 1973): 391-401.

[21] Phelps, Edmund S., and Pollak, Robert A. "On Second-Best National Saving and Game-Equilibrium Growth.' *Review of Economic Studies* 35 (April 1968): 185-99.

[22] Pollak, Robert A. "Consistent Planning.' *Review of Economic Studies* 35 (April 1968): 201-8.

[23] Harris, Christopher, Reny, Philip, and Robson, Arthur. "The Existence of Subgame Perfect Equilibrium in Continuous Games with Almost Perfect Information: a Case for Public Randomization", *Econometrica* 63 (May 1995): 507-544.

[24] Reny, Philip, and Robson, Arthur. "Existence of Subgame Perfect Equilibrium with Public Randomization: a Short Proof.' *Economics Bulletin* 3(24) (2002): 1-8.

[25] Strotz, Robert H. "Myopia and Inconsistency in Dynamic Utility Maximization.' *Review of Economic Studies* 23(3) (1956): 165-80.